Mathematical Practice

Introduction

The eight *Standards for Mathematical Practice* described in the Common Core State Standards [1] have opened up a great deal of discussion in K–12 mathematics. That's a good thing. Infusing school mathematics with an emphasis on what mathematics professionals do and think holds a great deal of promise for bringing some coherence and texture to the zoo of special purpose methods and tools that clutter our curricula and state tests. Simply raising the idea that there *is* a practice of mathematics—just as there's a practice of medicine or teaching—and then detailing some facets of that practice is so very refreshing.

My work and that of my colleagues at EDC has always put mathematical thinking—the habits of mind that are indigenous to our discipline—at the core of our work with teachers. What we've learned from expert teachers (see [2], for example) has led me to think more carefully about what it means to "work like a mathematician." Here are some of the things I've learned.

It's always been implicit in the design of the tasks and problems that we use in our professional development work (see [4], for example) that the problems provide a context in which one can employ certain mathematical practices. But I get the impression that the tasks themselves are often taken to codify (and sometimes define) that practice. So,

> **Thing 1:** *Mathematical practice lives in the approach to a task, not in the task. Rather than saying "here's a problem that exhibits this standard" we should be*

saying "Here's a problem where it's useful to employ this valuable way of thinking as a part of its solution."

These standards are supposed to exemplify the ways that proficient users of mathematics work. They are pillars supporting a whole practice rather than a taxonomy that covers the whole field.

Hy Bass puts it this way: "It will be helpful to name and (at least partially) specify some of the things—practices, dispositions, sensibilities, habits of mind—entailed in doing mathematics . . . *These are things that mathematicians typically do when they do mathematics. At the same time, most of these things, suitably interpreted or adapted, could apply usefully to elementary mathematics no less than to research*." [3]

> **Thing 2:** *The eight standards for mathematical practice don't comprise a discrete catalogue of this practice. Rather, they are important aspects, accentuating a much bigger style of work and an interconnected web of mathematical habits of mind used by mathematicians in their own work.*

Another slippery slope is to try to fit certain methods or approaches into one of eight buckets. In real mathematical practice, it is rare that a piece of work employs only one of those aspects of mathematical thinking described in the eight standards. Certainly, there are approaches to a particular problem that lean heavily on, say, exploiting mathematical structure (I'll give some examples below). But this hardly ever appears in isolation from other important ways of thinking used in mathematics.

> **Thing 3:** *The eight standards for mathematical practice describe a web of habits and dispositions that are used in tandem in real mathematical work. It's less important to identify a method as "a 4" than it is to help students become habituated to a style of work that makes use of several aspects of mathematical practice at once.*

Let me illustrate with three examples. Each of these examples starts with a pretty mundane task, and each of these tasks can be approached with many different methods. I want to illustrate specific methods that exemplify mathematical practice. The first two point out how two specific habits of mind are used. The last one shows how they all come together "in practice."

Unfortunately, many of the methods recommended in textbooks have nothing to do with mathematical practice—rather, they set up schoolish nonsense that works for the problem at hand but for no other problem in the universe.

Example 1: The dreaded algebra word problem

Here's the kind of thing we still use to torture kids:

There's actually some value to this kind of problem, if it is used as a device to help students learn the value of abstracting from numericals.

> Mary drives from Boston to Chicago, travels at an average rate of 60 MPH on the way down and 50 MPH on the way back. The total driving time takes 36 hours, how far is Boston from Chicago?

Every teacher knows that the difficult part here is coming up with the equation—solving it is another (often easier) matter. Rather than setting up boxes or using formulas, here's a method that builds on students' ability to solve similar problems in middle school: Take a guess. This is *not* to guess-and-check, getting closer to the answer each time. Remember—we want an equation, not a number.

So, think of the answer to this problem as an *equation* rather than a number.

- I take a guess, say 1200 miles.
- Check it:

  (∗) $\frac{1200}{60} = 20$

  (∗) $\frac{1200}{50} = 24$

  (∗) $20 + 24 \neq 36$
- That wasn't right, but that's okay— just keep track of your steps.
- Take another guess, say 1000, and check it, making the check more efficient:
$$\frac{1000}{60} + \frac{1000}{50} \stackrel{?}{=} 36$$
- Keep it up, until you get a "guess checker"
$$\frac{\text{guess}}{60} + \frac{\text{guess}}{50} \stackrel{?}{=} 36$$
- The equation is
$$\frac{x}{60} + \frac{x}{50} = 36$$

Of course, students need to understand already the relationship between speed, time, and distance—this kind of thing is described in the middle school content standards. Teaching it while trying to come up with an equation is a recipe for disaster.

This method for finding an equation that models a situation builds on students' ability to do numerical calculations that lie behind pre-algebra problems. The idea is to check enough guesses so that you get the regularity of the calculations needed to check the guesses; a generic "guess checker" is the desired equation. So, it involves:

1. working through several specific examples, concentrating on the rhythm of the calculations, and then

2. expressing this regularity in precise mathematical language.

The mathematical habits employed here are

1. abstract regularity from repeated calculations, and
2. use precise language (and algebraic symbolism) to give a generic and general description—the equation—for how you check your guesses.

Both of these are enshrined in Common Core (they are numbered 8 and 6 in case anyone cares).

The method described here captures some very common habits that are useful throughout algebra:

- carry out several concrete examples of a process that you don't quite "have in your head" in order to find regularity, and
- shoehorn that regularity into precise mathematical language to build a generic algorithm that describes every instance of the calculation.

If the only use of this technique were to set up contrived story problems, it would be as useless as the "box method" and its cousins that appear in texts. But I hope readers see that this method has legs. I use it all the time in my own mathematical work—I remember sitting up half the night performing the same calculation over and over with different inputs, struggling to organize what I felt in my gut into a precise formulation.

Example 2: Factoring

Factoring has gotten a bad name in recent years. That's not a surprise, considering the myriad of special-purpose methods that have appeared in curricula. But polynomial algebra sits at the historical core of algebra, and it plays a central role in current research.

Some factoring methods that can be introduced in elementary algebra *are* extensible, preview important ideas, and give students a chance to develop habits that let them transform an algebraic expression into one that reveals hidden meaning.

Because

$$(x+\alpha)(x+\beta) = x^2 + (\alpha+\beta)x + \alpha\beta,$$

factoring a quadratic polynomial $x^2 + bx + c$ (with leading coefficient 1) amounts to finding numbers α and β such that

$$\begin{aligned} \alpha+\beta &= b \quad \text{and} \\ \alpha\beta &= c \end{aligned}$$

This "sum-product" approach is fairly tractable for beginning students. To factor $x^2+14x+48$, students look for two numbers that add to 14 and multiply to 48, so that

$$x^2 + 14x + 48 = (x+6)(x+8)$$

Many students who can factor monic quadratics by the sum-product approach often have much more difficulty when the leading coefficient is not 1. Here again, there are general purpose methods that live on beyond their utility for developing this particular skill. One such method starts with the observation that $4x^2+36x+45$ can more easily be factored if one "chunks" the terms and writes it as

$$(2x)^2 + 18(2x) + 45$$

One can think of this as a "quadratic in $2x$," thinking of $2x$ as the variable. One can even replace $2x$ by some symbol, say z, and write the quadratic as

$$z^2 + 18z + 45$$

This factors by the sum-product method:

$$(z+15)(z+3)$$

Replacing z by $2x$ gives the factorization of the original quadratic.

The coefficients in this example was especially rigged for this technique. What if we are faced with something like

$$6x^2 + 11x - 10\ ?$$

One can reason like this:

1. Multiply the polynomial by 6 to make the leading coefficient a perfect square, remembering that we have to divide by 6 at some point to get back to where we started.

$$6\left(6x^2 + 11x - 10\right) = (6x)^2 + 11(6x) - 60$$

2. This is a quadratic in $6x$; let $z = 6x$, so the right-hand side becomes monic

$$z^2 + 11z - 60$$

3. This factors by the sum-product method

$$(z + 15)(z - 4)$$

4. But $z = 6x$, so we have

$$(6x + 15)(6x - 4)$$

5. Factor out common factors—3 from the first binomial and 2 from the second, producing

$$6(2x + 5)(3x - 2)$$

6. Dividing by 6 gives the factorization of the original polynomial.

This "scaling method" is a general-purpose tool that has applications all over algebra and calculus—it amounts to a change of variable in order to hide complexity. Our use of this technique extends to a method that applies to polynomials of any degree, allowing one to transform a polynomial in one variable into one that is monic, one of the steps used to derive Cardano's algorithm for cubics.

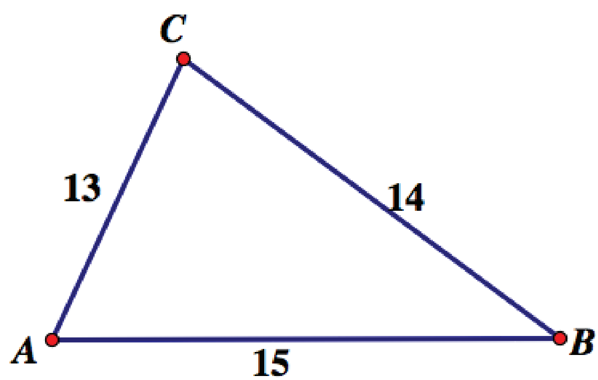
Notice that the insight here did not come from abstracting regularity from numerical examples; rather it came from seeing a certain *structure* in the quadratic polynomial. This kind of seeing and using structure in expressions is at the heart of another standard for mathematical practice (the famous number 7).

In a sense, all of mathematics is about structure. Common Core's use of structure is concerned with two facets of mathematical structure: structure in numerical and algebraic *expressions* and structure in mathematical *systems* ($\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$, for example).

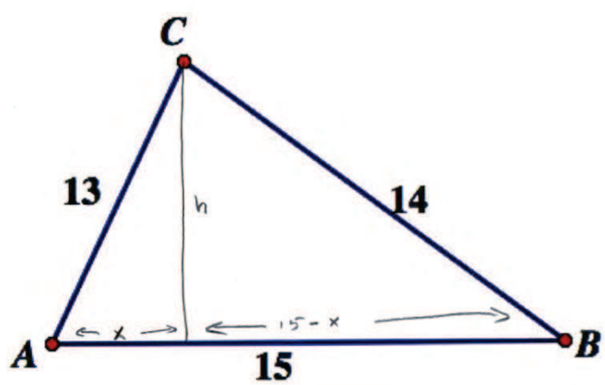
Example 3: Heron's Formula

A triangle is determined by the lengths of its sides—this is the SSS theorem in geometry. Hence, one should be able to figure out the area of a triangle from its three side-lengths. Heron's formula gives exactly this, but, if it is treated at all in geometry curricula, it's pulled out like a rabbit from a hat, with no justification. But somebody (maybe Heron) had to have figured it out for the first time, not with the aid of a rabbit or textbook. It's good for kids to know that they can figure out things they haven't been taught, too. Here's one approach to the formula that makes use of several aspects for mathematical practice at once.

First of all, if I'm in the habit of abstracting from numericals, I'll want to find the area of a *specific* triangle and see if I can do it in a way that doesn't depend on the actual numbers. So start with a carefully crafted triangle:



I know that the area is $\frac{1}{2}$ the product of the base and the height, so I draw in a height and label some new pieces:



Even here, if I'm attuned to abstracting regularity, I see a method lurking that will work for any numerical example. Plowing on, I set up two equations, using the Pythagorean theorem:

$$x^2 + h^2 = 169$$
$$(15 - x)^2 + h^2 = 196$$

Expand the second equation:

$$225 - 30x + x^2 + h^2 = 196$$

Ah, but $x^2 + h^2 = 169$ from the first equation, so

$$225 - 30x + 169 = 196$$

From here, you can find x, then h, and then finally the area.

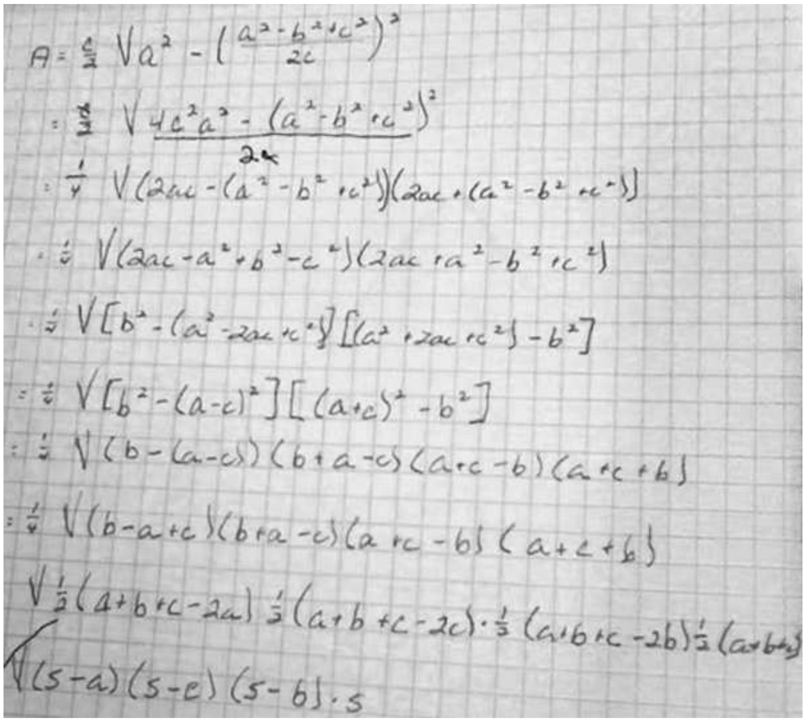
If I were doing this for real, I'd try it with some other triangles—using not-so-nice numbers—trying to get the rhythm of the method and then forcing it into precise language. Even in this one example, if you stare long enough, you can see the role played by 13, 14, and 15 in the calculations, especially of you "delay the evaluation" and write the last equation as

$$15^2 - 2 \cdot 15x + 13^2 = 14^2$$

This delayed evaluation is one way to reveal hidden structure (another standard for mathematical practice). And, if the triangle had side-lengths a, b, and c, you'd end up with

$$c^2 - 2cx + a^2 = b^2$$

Here's where the structure of the expressions becomes dominant, and by careful (and mindful) transformations, you can use this to carry out a generic version of your numerical calculations and transform them into a derivation of Heron's formula, as this teacher did:



It's impossible to put this approach into one of the eight buckets. It employs in inextricable ways abstracting regularity, using precision, and exploiting structure, all at once. Sometimes one habit becomes more dominant (as in the final transformations), but they are all here, all the time. As an exercise, you might try using this method to derive a formula for the area using other pieces that determine a triangle (the three altitudes or medians, for example).

These examples are typical of how the practice of mathematics is exercised in nature. Imagine a K–12 mathematics program that elevates this practice to the same levels of importance as technical fluency and factual knowledge. Such a program would be easier to teach and easier for students to understand. And it might just purge school mathematics of all the exotic paraphernalia that we force on so many kids, making school mathematics align with how mathematics is practiced outside of school.

Paul Goldenberg contributed to every aspect of this post, from the ideas to the editing.

Bibliography

[1] Common Core State Standards for Mathematics: http://www.corestandards.org/Math/Practice/

[2] PROMYS for Teachers: https://promys.org/pft

[3] "A Vignette of Doing Mathematics." *The Montana Mathematics Enthusiast*, 2011.

[4] Implementing the Mathematical Practice Standards: http://mathpractices.edc.org/