Consider the set $\mathcal{M}$ of all Riemannian metrics with the topology of a sphere in 2D. It is natural to ask whether one could endow this set with a natural probability measure, i.e., construct some uniform probability measure on the space of all Riemannian spheres. This question is not only mathematically appealing, but has its roots in physics—it gives rise to a 2D model-case of quantum gravity, called Liouville quantum gravity.

There has been tremendous recent progress in giving sense to a natural probability measure on $\mathcal{M}$ from two different directions. In the first program, one discretizes the problem using certain tilings of the sphere, called random planar maps. On each level of discretization, one can define an approximation of the uniform metric on the sphere, and then take a limit. It has been shown that the limiting object, called the Brownian map or the Brownian sphere, exists, and although it is too rough to live in $\mathcal{M}$, it gives rise to a probability measure on topological spheres endowed with an area measure and a metric [J.-F. Le Gall, Invent. Math. 169 (2007), no. 3, 621–670; MR2336042; G. Miermont, Acta Math. 210 (2013), no. 2, 319–401; MR3070569]. In the second program, one starts from a different simplification: as all 2D Riemannian metrics come with a unique conformal structure, one can parametrize the whole space of Riemannian metrics on the 2D sphere via metrics of the form $\exp(\gamma h)d\rho$ on the Riemannian sphere $S^2$. Thereafter, one can try to construct a probability measure on solely the conformal factors of the form $\exp(\gamma h)$ [B. Duplantier and S. Sheffield, Invent. Math. 185 (2011), no. 2, 333–393; MR2819163]. The current article is the first article in a three-article series [“Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding”, preprint, arXiv:1605.03563; “Liouville quantum gravity and the Brownian map III: the conformal structure is determined”, preprint, arXiv:1608.05391] by the same authors, establishing the equivalence of these two directions, i.e., proving that the two directions give rise to a unique probability law on metric-measure spaces with the topology of a sphere, equipped with a conformal structure.

A major obstacle in proving the equivalence of two directions is the fact that in the first program the limiting random object comes with an area measure and a metric, but without a conformal structure, i.e., without a natural parametrization of the object by $S^2$. In the second program, however, there is a natural parametrization by $S^2$ and it is simple to construct a random area measure, but the difficult part is constructing the random metric. The current article basically provides a construction of the missing metric in the second program, and by the end of their three-article series, the authors establish that in fact both programs encode an area measure, a metric and a conformal structure and that the two triplets of the two programs agree. It should be noted that by now a second approach to constructing metrics, encompassing a larger family of models, has also appeared [E. Gwynne and J. Miller, “Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0,2)$”, preprint, arXiv:1905.00383, Invent. Math., to appear], yet it builds a link to random planar maps only through this three-article series.

Let us describe in a few words the approach taken by the authors to construct this random metric. The starting point is $S^2$ together with a random area law $\mu$. 
In an earlier work [Duke Math. J. 165 (2016), no. 17, 3241–3378; MR3572845], the same authors constructed a random growth process, called quantum Loewner evolution (QLE), running on top of \( \mu \) and proposed it as a candidate for describing the growth of a metric ball away from a given point. It is then natural to try to define the distance between two points \( x \) and \( y \) by just taking one of the points, say \( x \), and running this QLE growth process \( Q_{x \to y}(t) \) until the other point is reached. If the parametrization is judiciously chosen, the total time is a good candidate for the distance. However, leaving aside the question of a judicious parametrization, there are several serious problems: firstly, this growth process \( Q_{x \to y} \) is a priori not measurable with respect to the area law \( \mu \), and thus it is not clear that one would get the same distance when one takes different samples of this growth process; second, one could as well start the growth process \( Q_{y \to x}(t) \) from the point \( y \) until it reaches \( x \) and a priori there is no reason that the total time should agree.

The key mathematical content of the article is to cleverly overcome these problems by establishing a certain symmetry (Section 7). Instead of running \( Q_{x \to y}(t) \) from \( x \) until it reaches \( y \), one grows the metric ball around \( x \) until a uniform random time \( \tau \) from the total time length, and then runs an independent process \( Q_{x \to y}(s) \) from \( y \) until it touches \( Q_{x \to y}(\tau) \). Roughly put, the authors prove that the resulting picture is symmetric—one cannot tell the difference between which of the metric balls was grown first. In fact, the symmetry statement is set up in such a slick way that establishing it proves rather directly that the distances are well defined and that they are symmetric, satisfy the triangle inequality and are determined by the area measure \( \mu \) (Section 8). The symmetry itself is established in two steps. A certain global symmetry can be deduced from a corresponding symmetry for SLE\( _6 \) growth processes (Section 5), combined with the fact that QLE can be well approximated by certain reshuffled versions of these SLE\( _6 \) processes (Section 6). Thereafter, it remains to establish a local symmetry around the meeting point, which is established using further local reshuflings (Section 7).

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.

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