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Liouville quantum gravity and the Brownian map I: the $\mathrm{QLE}(8/3,0)$ **metric. (English summary)**
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Consider the set $\mathcal{M}$ of all Riemannian metrics with the topology of a sphere in 2D. It is natural to ask whether one could endow this set with a natural probability measure, i.e., construct some uniform probability measure on the space of all Riemannian spheres. This question is not only mathematically appealing, but has its roots in physics—it gives rise to a 2D model-case of quantum gravity, called Liouville quantum gravity.

There has been tremendous recent progress in giving sense to a natural probability measure on $\mathcal{M}$ from two different directions. In the first program, one discretizes the problem using certain tilings of the sphere, called random planar maps. On each level of discretization, one can define an approximation of the uniform metric on the sphere, and then take a limit. It has been shown that the limiting object, called the Brownian map or the Brownian sphere, exists, and although it is too rough to live in $\mathcal{M}$, it gives rise to a probability measure on topological spheres endowed with an area measure and a metric [J.-F. Le Gall, Invent. Math. **169** (2007), no. 3, 621–670; MR2336042; G. Miermont, Acta Math. **210** (2013), no. 2, 319–401; MR3070569]. In the second program, one starts from a different simplification: as all 2D Riemannian metrics come with a unique conformal structure, one can parametrize the whole space of Riemannian metrics on the 2D sphere via metrics of the form $\exp(\gamma h)d\rho$ on the Riemannian sphere S^2. Thereafter, one can try to construct a probability measure on solely the conformal factors of the form $\exp(\gamma h)$ [B. Duplantier and S. Sheffield, Invent. Math. **185** (2011), no. 2, 333–393; MR2819163]. The current article is the first article in a three-article series ["Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding", preprint, arXiv:1605.03563; "Liouville quantum gravity and the Brownian map III: the conformal structure is determined", preprint, arXiv:1608.05391] by the same authors, establishing the equivalence of these two directions, i.e., proving that the two directions give rise to a unique probability law on metric-measure spaces with the topology of a sphere, equipped with a conformal structure.

A major obstacle in proving the equivalence of two directions is the fact that in the first program the limiting random object comes with an area measure and a metric, but without a conformal structure, i.e., without a natural parametrization of the object by S^2. In the second program, however, there is a natural parametrization by S^2 and it is simple to construct a random area measure, but the difficult part is constructing the random metric. The current article basically provides a construction of the missing metric in the second program, and by the end of their three-article series, the authors establish that in fact both programs encode an area measure, a metric and a conformal structure and that the two triplets of the two programs agree. It should be noted that by now a second approach to constructing metrics, encompassing a larger family of models, has also appeared [E. Gwynne and J. Miller, "Existence and uniqueness of the Liouville quantum gravity metric for $\gamma \in (0,2)$", preprint, arXiv:1905.00383, Invent. Math., to appear], yet it builds a link to random planar maps only through this three-article series.

Let us describe in a few words the approach taken by the authors to construct this random metric. The starting point is S^2 together with a random area law μ.

In an earlier work [Duke Math. J. **165** (2016), no. 17, 3241–3378; MR3572845], the same authors constructed a random growth process, called quantum Loewner evolution (QLE), running on top of μ and proposed it as a candidate for describing the growth of a metric ball away from a given point. It is then natural to try to define the distance between two points x and y by just taking one of the points, say x, and running this QLE growth process $Q_{x\to y}(t)$ until the other point is reached. If the parametrization is judiciously chosen, the total time is a good candidate for the distance. However, leaving aside the question of a judicious parametrization, there are several serious problems: firstly, this growth process $Q_{x\to y}$ is a priori not measurable with respect to the area law μ, and thus it is not clear that one would get the same distance when one takes different samples of this growth process; second, one could as well start the growth process $Q_{y\to x}(t)$ from the point y until it reaches x and a priori there is no reason that the total time should agree.

The key mathematical content of the article is to cleverly overcome these problems by establishing a certain symmetry (Section 7). Instead of running $Q_{x\to y}(t)$ from x until it reaches y, one grows the metric ball around x until a uniform random time τ from the total time length, and then runs an independent process $Q_{y\to x}(s)$ from y until it touches $Q_{x\to y}(\tau)$. Roughly put, the authors prove that the resulting picture is symmetric—one cannot tell the difference between which of the metric balls was grown first. In fact, the symmetry statement is set up in such a slick way that establishing it proves rather directly that the distances are well defined and that they are symmetric, satisfy the triangle inequality and are determined by the area measure μ (Section 8). The symmetry itself is established in two steps. A certain global symmetry can be deduced from a corresponding symmetry for SLE_6 growth processes (Section 5), combined with the fact that QLE can be well approximated by certain reshuffled versions of these SLE_6 processes (Section 6). Thereafter, it remains to establish a local symmetry around the meeting point, which is established using further local reshufflings (Section 7).

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References

1. Aru, J., Huang, Y., Sun, X.: Two perspectives of the 2D unit area quantum sphere and their equivalence. Commun. Math. Phys. **356**(1), 261–283 (2017). arXiv:1512.06190 MR3694028
2. Bertoin, J.: Lévy Processes, Volume 121 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge (1996) MR1406564
3. Bishop, C.J.: Conformal welding and Koebe's theorem. Ann. Math. (2) **166**(3), 613–656 (2007) MR2373370
4. Curien, N., Le Gall, J.-F.: The Brownian plane. J. Theor. Probab. **27**(4), 1249–1291 (2014). arXiv:1204.5921 MR3278940
5. Curien, N., Le Gall, J.-F.: The hull process of the Brownian plane. Probab. Theory Related Fields **166**(1–2), 187–231 (2016). arXiv:1409.4026 MR3547738
6. Curien, N., Le Gall, J.-F.: Scaling limits for the peeling process on random maps. Ann. Inst. Henri Poincaré Probab. Stat. **53**(1), 322–357 (2017). arXiv:1412.5509 MR3606744
7. Chassaing, P., Schaeffer, G.: Random planar lattices and integrated super Brownian excursion. Probab. Theory Related Fields **128**(2), 161–212 (2004). arXiv:math/0205226 MR2031225
8. Cori, R., Vauquelin, B.: Planar maps are well labeled trees. Can. J. Math. **33**(5), 1023–1042 (1981) MR0638363
9. David, F., Kupiainen, A., Rhodes, R., Vargas, V.: Liouville quantum gravity on the

Riemann sphere. Commun. Math. Phys. **342**(3), 869–907 (2016). arXiv:1410.7318 MR3465434

10. Duplantier, B., Miller, J., Sheffield, S.: Liouville quantum gravity as a mating of trees. ArXiv e-prints (Sept. 2014). arXiv:1409.7055 MR3729063
11. David, F., Rhodes, R., Vargas, V.: Liouville quantum gravity on complex tori. J. Math. Phys. **57**(2), 022302, 25 (2016). arXiv:1504.00625 MR3450564
12. Duplantier, B., Sheffield, S.: Liouville quantum gravity and KPZ. Invent. Math. **185**(2), 333–393 (2011). arXiv:0808.1560 MR2819163
13. Gwynne, E., Miller, J.: Convergence of the topology of critical Fortuin–Kasteleyn planar maps to that of CLE_κ on a Liouville quantum surface (**in preparation**) (2016)
14. Gwynne, E., Sun, X.: Scaling limits for the critical Fortuin–Kastelyn model on a random planar map III: finite volume case. ArXiv e-prints (Oct. 2015). arXiv:1510.06346 MR3661659
15. Gwynne, E., Sun, X.: Scaling limits for the critical Fortuin–Kasteleyn model on a random planar map II: local estimates and empty reduced word exponent. Electron. J. Probab. **22**(45), 56 (2017). arXiv:1505.03375 MR3661659
16. Jones, P.W., Smirnov, S.K.: Removability theorems for Sobolev functions and quasiconformal maps. Ark. Mat. **38**(2), 263–279 (2000) MR1785402
17. Krikun, M.: Uniform infinite planar triangulation and related time-reversed critical branching process. J. Math. Sci. **131**(2), 5520–5537 (2005) MR2050691
18. Kyprianou, A.E.: Introductory Lectures on Fluctuations of Lévy Processes with Applications. Universitext. Springer, Berlin (2006) MR2250061
19. Lamperti, J.: Continuous state branching processes. Bull. Am. Math. Soc. **73**, 382–386 (1967) MR0208685
20. Le Gall, J.-F.: Spatial Branching Processes, Random Snakes and Partial Differential Equations. Lectures in Mathematics ETH Zürich. Birkhäuser, Basel (1999) MR1714707
21. Le Gall, J.-F.: Uniqueness and universality of the Brownian map. Ann. Probab. **41**(4), 2880–2960 (2013). arXiv:1105.4842 MR3112934
22. Le Gall, J.-F.: The Brownian map: a universal limit for random planar maps. In: XVIIth International Congress on Mathematical Physics, pp. 420–428. World Sci. Publ., Hackensack (2014) MR3204495
23. Le Gall, J.-F., Paulin, F.: Scaling limits of bipartite planar maps are homeomorphic to the 2-sphere. Geom. Funct. Anal. **18**(3), 893–918 (2008). arXiv:math/0612315 MR2438999
24. Miermont, G.: The Brownian map is the scaling limit of uniform random plane quadrangulations. Acta Math. **210**(2), 319–401 (2013) MR3070569
25. Marckert, J.-F., Mokkadem, A.: Limit of normalized quadrangulations: the Brownian map. Ann. Probab. **34**(6), 2144–2202 (2006) MR2294979
26. Miller, J., Sheffield, S.: An axiomatic characterization of the Brownian map. ArXiv e-prints (June 2015). arXiv:1506.03806 MR4225028
27. Miller, J., Sheffield, S.: Liouville quantum gravity spheres as matings of finite-diameter trees. ArXiv e-prints (June 2015) arXiv:1506.03804. To appear in Annales de l'Institut Henri Poincaré MR4010949
28. Miller, J., Sheffield, S.: Liouville quantum gravity and the Brownian map II: geodesics and continuity of the embedding. ArXiv e-prints (May 2016). arXiv:1605.03563 MR4242633
29. Miller, J., Sheffield, S.: Liouville quantum gravity and the Brownian map III: the conformal structure is determined. ArXiv e-prints (Aug. 2016). arXiv:1608.05391 MR4242633

30. Miller, J., Sheffield, S.: Imaginary geometry I: interacting SLEs. Probab. Theory Related Fields **164**(3–4), 553–705 (2016). arXiv:1201.1496 MR3477777
31. Miller, J., Sheffield, S.: Imaginary geometry II: reversibility of $\mathrm{SLE}_\kappa(\rho_1;\rho_2)$ for $\kappa \in (0,4)$. Ann. Probab. **44**(3), 1647–1722 (2016). arXiv:1201.1497 MR3502592
32. Miller, J., Sheffield, S.: Imaginary geometry III: reversibility of SLE_κ for $\kappa \in (4,8)$. Ann. Math. (2) **184**(2), 455–486 (2016). arXiv:1201.1498 MR3548530
33. Miller, J., Sheffield, S.: Quantum Loewner evolution. Duke Math. J. **165**(17), 3241–3378 (2016). arXiv:1312.5745 MR3572845
34. Miller, J., Sheffield, S.: Imaginary geometry IV: interior rays, whole-plane reversibility, and space-filling trees. Probab. Theory Related Fields **169**(3–4), 729–869 (2017). arXiv:1302.4738 MR3719057
35. Miller, J., Wu, H.: Intersections of SLE paths: the double and cut point dimension of SLE. Probab. Theory Related Fields **167**(1–2), 45–105 (2017). arXiv:1303.4725 MR3602842
36. Polyakov, A.M.: Quantum geometry of bosonic strings. Phys. Lett. B **103**(3), 207–210 (1981) MR0623209
37. Polyakov, A.M.: Quantum geometry of fermionic strings. Phys. Lett. B **103**(3), 211–213 (1981) MR0623210
38. Polyakov, A.: Two-dimensional quantum gravity. Superconductivity at high T_C. Champs. cordes et phénomènes critiques (Les Houches, 1988), pp. 305–368. North-Holland, Amsterdam (1990) MR1052937
39. Rohde, S., Schramm, O.: Basic properties of SLE. Ann. Math. (2) **161**(2), 883–924 (2005) MR2153402
40. Schaeffer, G.: Conjugaison d'arbres et cartes combinatoires aléatoires. Ph.D. Thesis, Université Bordeaux I (1998)
41. Sheffield, S.: Conformalweldings of random surfaces: SLE and the quantum gravity zipper. Ann. Probab. **44**(5), 3474–3545 (2016). arXiv:1012.4797 MR3551203
42. Sheffield, S.: Quantum gravity and inventory accumulation. Ann. Probab. **44**(6), 3804–3848 (2016). arXiv:1108.2241 MR3572324
43. Schramm, O., Sheffield, S.: A contour line of the continuum Gaussian free field. Probab. Theory Related Fields **157**(1–2), 47–80 (2013). arXiv:1008.2447 MR3101840
44. Tutte, W.T.: A census of planar maps. Can. J. Math. **15**, 249–271 (1963) MR0146823

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.