MR3993416 35R11 35A15 35B33 47G20
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Superlinear Schrödinger-Kirchhoff type problems involving the fractional p-Laplacian and critical exponent. (English summary)

Adv. Nonlinear Anal. **9** (2020), *no. 1,* 690–709.

In this paper the authors obtain existence and multiplicity results for the following class of p-fractional Kirchhoff-type problems with a critical exponent:

$$M(\|u\|_\lambda^p)\left[\lambda(-\Delta)_p^s u+V(x)|u|^{p-2}u\right]=|u|^{p_s^*-2}u+f(x,u)\quad \text{in } \mathbb{R}^N, \tag{1}$$

where $(-\Delta)_p^s$ is the fractional p-Laplace operator with $0<s<1<p<\frac{N}{s}$, $p_s^*=\frac{Np}{N-sp}$ is the critical fractional exponent, $\lambda>0$ is a parameter, and $f\colon\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function. Here,

$$\|u\|_\lambda=(\lambda[u]_{s,p}^p+\|u\|_{p,V}^p)^{\frac{1}{p}},$$

where

$$[u]_{s,p}^p=\iint_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}dxdy$$

and

$$\|u\|_{p,V}^p=\int_{\mathbb{R}^N}V(x)|u|^p dx,$$

and the functions M and V satisfy the following assumptions:

(M) $M\in C(\mathbb{R},\mathbb{R})$ and there exist $\theta\in(1,\frac{p_s^*}{p}]$ and $0<m_0\le m_1$ such that

$$m_0t^{\theta-1}\le M(t)\le m_1t^{\theta-1}\quad \text{for all } t\in\mathbb{R}_0^+;$$

(V) $V\in C(\mathbb{R}^N,\mathbb{R})$, $V(x_0)=\min_{x\in\mathbb{R}^N}V(x)=0$ and there exists $h>0$ such that the Lebesgue measure of $\{x\in\mathbb{R}^N: V(x)<h\}$ is finite; there exists $\rho>0$ such that $\lim_{|y|\to\infty}\operatorname{meas}(\{x\in B_\rho(y): V(x)<c\})=0$ for any $c\in\mathbb{R}^+$.

Firstly, when $\theta\in(1,\frac{p_s^*}{p})$ in (M) and f satisfies the following conditions:

(f_1) there exists $q\in(\theta p,p_s^*)$ such that for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$|f(x,t)|\le\theta p\varepsilon|t|^{\theta p-1}+qC_\varepsilon|t|^{q-1}\quad \text{for a.e. } x\in\mathbb{R}^N \text{ and all } t\in\mathbb{R},$$

(f_2) there exists $q_1>\frac{m_1\theta p}{m_0}$ such that

$$q_1F(x,t)\le f(x,t)t\quad \text{for all } (x,t)\in\mathbb{R}^N\times\mathbb{R},$$

(f_3) there exists $q_2\in(\theta p,p_s^*)$ such that $F(x,t)\ge a_0|t|^{q_2}$ for a.e. $x\in\mathbb{R}^N$ and all $t\in\mathbb{R}$,

the authors use the mountain pass lemma, a concentration-compactness argument, and index theory to prove that for any $\lambda>0$ there exists $\lambda^*>0$ such that (1) has a nontrivial solution u_λ for any $\lambda\in(0,\lambda^*)$ which satisfies

$$\|u_\lambda\|_\lambda^p\le\left(\frac{\theta pq_1}{m_0q_1-m_1\theta p}\right)^{\frac{1}{\theta}}\sigma^{\frac{1}{\theta}}\lambda^{\frac{p_s^*}{p_s^*-\theta p}}, \tag{2}$$

where $\sigma=\frac{1}{q_1}(1-\frac{m_1}{m_0})+\frac{1}{\theta p}-\frac{1}{p_s^*}$. In addition, if $f(x,t)$ is odd with respect to t, then for any $m\in\mathbb{N}$ there exists $\lambda_m>0$ such that (1) admits at least m pairs of solutions $u_{\lambda,i}$

$(i=1,\dots,m)$ which satisfy (2) whenever $\lambda\in(0,\lambda_m]$.

Secondly, when $\theta=\frac{p_s^*}{p}$ in (M), $2\le p<\frac{N}{s}$, $f(x,t)$ is odd with respect to t and satisfies the following conditions:

(f_4) there exists $q\in(p,p_s^*)$ such that for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such that

$$|f(x,t)|\le p\varepsilon|t|^{\theta p-1}+qC_\varepsilon|t|^{q-1}\ \text{ for a.e. } x\in\mathbb{R}^N \text{ and all } t\in\mathbb{R},$$

(f_5) there exists $q_1\in(\theta p,p_s^*)$ such that $F(x,t)\ge a_0|t|^{q_1}$ for a.e. $x\in\mathbb{R}^N$ and all $t\in\mathbb{R}$,

the authors use Krasnosel′skiĭ's genus theory and Clark's theorem to deduce that (1) has infinitely many pairs of distinct solutions for any $\lambda>2^pS^{-\frac{p_s^*}{p}}/m_0$, where $S>0$ is defined as

$$S=\inf_{u\in D^{s,p}(\mathbb{R}^N)\setminus\{0\}}\frac{[u]_{s,p}^p}{|u|_{p_s^*}^p}.$$

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