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★**A comprehensive introduction to sub-Riemannian geometry.**
From the Hamiltonian viewpoint.
With an appendix by Igor Zelenko.
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Sub-Riemannian geometry is the geometry of a world with nonholonomic constraints. In such a world, one can move, send and receive information only in certain admissible directions but eventually one can reach every position from any other. In the last two decades sub-Riemannian geometry has emerged as an independent research domain impacting on several areas of pure and applied mathematics, including control theory, harmonic and complex analysis, subelliptic PDEs, geometric measure theory, calculus of variations, optimal transport, and potential analysis.

This book originated from a 3-semester-long course given by A. Agrachev in SISSA (Trieste, Italy), in the years 2008/09, 2009/10 and 2010/11. It grew steadily in content, depth and generality over the course of the subsequent 10 years. Its final published form, an impressive ∼800 page volume, features a modern and complete introduction to sub-Riemannian geometry, from the point of view of geometrical control theory. The authors, all well-established experts in the field, personally contributed to the development of several of the topics treated in the text.

A "live" version of the book has been available online in the course of its development, giving it a certain visibility before its publication. It was used as the basis for several courses, at different levels and universities or schools. This monograph is set to become one of the main references in the field, together with the pioneering book edited by A. Bellaïche and J.-J. Risler [*Sub-Riemannian geometry*, Progr. Math., 144, Birkhäuser, Basel, 1996; MR1421821] and the classical one by R. W. Montgomery [*A tour of subriemannian geometries, their geodesics and applications*, Math. Surveys Monogr., 91, Amer. Math. Soc., Providence, RI, 2002; MR1867362].

The title of the book is humble: in my opinion this is not only a *comprehensive introduction* to sub-Riemannian geometry, but an in-depth and complete textbook, ranging from basic topics (suitable, e.g., for an undergraduate course) to advanced research ones, useful for graduate students and researchers alike. The book can be divided into four parts: (I) basic theory, (II) core topics, (III) curvature, and (IV) a selection of research topics.

I. Basic theory (Chapters 1–5).

This part of the book contains the basic definitions and results, which are presented from a very general and flexible viewpoint. The main results proved here are the finiteness and continuity of the sub-Riemannian distance, the existence of length-minimizers, the infinitesimal characterization of geodesics, the Hamiltonian flow and its properties. Chapter 1, about classical geometry of surfaces in $\mathbb{R}^3$, might seem out of place, but it is actually useful to get a feeling for the notation and the geometric control theory approach that the authors employ throughout the book.

II. Core topics (Chapters 6–13).

This constitutes the hard core of the book. It starts with a thorough introduction to chronological calculus. This tool, powerful but not easy to master, allows one to work in an efficient way with flows of non-autonomous vector fields, and will be used on several occasions throughout the book. Chapters 7 and 9 are devoted to two main classes of examples: homogeneous sub-Riemannian structures on Lie groups (including the ubiquitous Heisenberg group), and almost-Riemannian structures in dimension two (e.g., the Grushin plane). Chapter 8 introduces two core objects in sub-Riemannian geometry: the end-point and the exponential maps, and studies their properties. Here, several key results about minimality properties of geodesics are extended from the Riemannian to the sub-Riemannian case, in the absence of non-trivial abnormal length-minimizers. Chapter 10 features a beautiful and original definition of the nonholonomic tangent space. It is intrinsic, and it is based on the concept of "admissible variations" instead of the usual privileged coordinates approach. Chapter 11 studies the regularity properties of the sub-Riemannian distance, which are strikingly different with respect to the Riemannian case. The authors prove in particular the celebrated result about the density of the set of smooth points of the sub-Riemannian distance. Chapter 12, one of the most technically demanding ones in the book, is devoted to the analysis of the second-order conditions for length-minimizers and the analysis of abnormal extremals. Chapter 13 features the sub-Riemannian optimal synthesis for some model spaces, and can be seen as an extension of Chapter 7. In particular, the authors make use of a technique to compute the cut locus (attributed originally to Hadamard for Riemannian surfaces), which is systematically presented here for the first time.

III. Curvature (Chapters 14–17 and Appendix).

This part contains a comprehensive presentation of curvature in sub-Riemannian geometry. This approach is based on the theory of curves in the Lagrange Grassmannian pioneered by Agrachev in the 1990s, and developed subsequently by several other authors. In this part we shall include the Appendix by I. Zelenko, on the construction of canonical frames for curves in the Lagrange Grassmannian, which is an important tool used in modern research in sub-Riemannian curvature and curvature bounds.

IV. Miscellanea (Chapters 18–21).

The final chapters gather a selection of different topics. Chapter 18 can be seen as a continuation of Chapter 5, and is devoted to the integrability of the sub-Riemannian geodesic flow on 3D Lie groups. Chapter 19 can be seen as an advanced continuation of Chapter 13. Here the authors study the small-distance asymptotics of the exponential map for the 3D contact case, giving information on the structure of the conjugate/cut locus and its relation with the curvature. Chapter 20 is devoted to the problem of volumes in sub-Riemannian geometry, in particular the Popp volume, the Hausdorff volume, their relations and their regularity. Chapter 21 contains an informal introduction to the sub-Laplacian, the corresponding heat equation, including the explicit formula for the heat kernel in the Heisenberg group. The latter is proved via an original method based on the standard Fourier transform. This final chapter perhaps lacks a systematic treatment of heat kernel bounds and asymptotics, even though the authors provide several references in the bibliographical notes.

What is not in the book:

Of course certain topics have been left out: geometric measure theory and calculus of variations on Carnot groups, hypoelliptic PDEs associated with sub-Riemannian structures (in particular on nilpotent groups), potential analysis, optimal transport, theory of distributions à la Cartan, interaction with physics and applications. Their absence should not be seen as a shortcoming, as there are already several excellent sources focusing on these specific subjects.

To conclude, this textbook is a valuable reference in sub-Riemannian geometry, providing a systematic and firm foundation to the theory. The subject is developed within the general framework of sub-Riemannian structures with possibly non-constant rank on arbitrary manifolds, that is at the greatest possible level of generality. The point of view is that of Hamiltonian geometry and geometric control theory. The exposition is clear and in the proofs much emphasis is given to the geometrical intuition and intrinsic arguments, rather than to local computations. The book covers a lot of material, and any course based on it should be an informative one. Some sections in the second half of part II might be quite demanding for students, even graduate ones, but this is the price to pay for depth and generality, and it is worth the effort. It is my opinion that this textbook will serve as a solid reference for many researchers in the field, and will contribute to the development of the subject in the forthcoming years. *Luca Rizzi*