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Iterating symmetric extensions. (English summary)
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In the paper under review, the author develops a technique for iterating symmetric extension models of ZF (i.e., Zermelo-Fraenkel set theory minus the axiom of choice (AC)) in order to construct new models of ZF. (A symmetric model of ZF is an intermediate model between a ground model M and a generic extension model $M[G]$ of M, which is determined by a group $\mathcal{G}$ of automorphisms of a forcing notion $(\mathbb{P}, \leq)$ and a normal filter $\mathcal{F}$ of subgroups over $\mathcal{G}$.) As the author mentions in his paper, precursors to the idea of iterating symmetric extensions can be seen in the works of G. Sageev [Ann. Math. Logic **8** (1975), 1–184; MR0366668; Ann. Math. Logic **21** (1981), no. 2-3, 221–281 (1982); MR0656794] and G. P. Monro [Fund. Math. **80** (1973), no. 2, 105–110; MR0347602]. However, in this reviewer's opinion, the paper under review can be considered as the first one which addresses in a most specific way the issue of iterating symmetric extensions.

The paper starts with a well-presented overview of iterated forcing and symmetric extensions and then delves into the development of the method, which allows the author to extend the general structure of symmetric extensions (that is, automorphisms and then filters of groups) to the iteration. He firstly investigates the extension of automorphisms for two-step iterations (and uses mixing in a substantial way when defining the iterated automorphisms—Proposition 3.1 of the paper) and then moves on to the general case, that is, to finite support iterations. A subtle issue is that the structure of the first symmetric extension acts on the second one and so forth, and the author carefully addresses these actions by establishing that in his technique the requirement on each step of the iteration being respected by the previous collection of automorphisms is a nontrivial requirement.

He then discusses the second crucial point, that is, the extension of filters. A central issue here is the preservation of the normality of the filters. In order to tackle this problem, the author makes some additional nontrivial assumptions for combining filters and the notion of supports, and introduces the notions of $\mathcal{F}_\delta$ support, excellent support, $\mathcal{F}_\delta$-respected name (and hereditarily $\mathcal{F}_\delta$-respected name), and for a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ ($\mathbb{P}$ a forcing notion, $\mathcal{G}$ a group of automorphisms of $\mathbb{P}$, and $\mathcal{F}$ a normal filter of subgroups over $\mathcal{G}$) the notion of $\mathbb{P}$ being $\mathcal{F}$-tenacious: a condition $p \in \mathbb{P}$ is $\mathcal{F}$-tenacious if and only if there exists $H \in \mathcal{F}$ which fixes p, i.e., for every $\pi \in H$, $\pi p = p$; $\mathbb{P}$ is $\mathcal{F}$-tenacious if and only if there is a dense subset of $\mathcal{F}$-tenacious conditions. The machinery developed here essentially aims to ensure that in some sense large groups can be conjugated and remain inside "the iteration of filters".

The author then continues by applying (in Section 5 of the paper) his methods of extending the general structure of symmetric models to the iteration to discuss the class of names and the forcing relation (IS-names and IS-forcing relation, respectively) identifying the intermediate model (this model, which lies between a ground model V and its generic extension model $V[G]$, is the author's class $\mathsf{IS}_\delta^G = \{\dot{x}^G \mid \dot{x} \in \mathsf{IS}_\delta\}$, where $G \subseteq \mathbb{P}_\delta$ is a V-generic filter and δ is the length of a symmetric iteration $\langle \dot{\mathbb{Q}}_\alpha, \dot{\mathcal{G}}_\alpha, \dot{\mathcal{F}}_\alpha \mid \alpha < \delta \rangle$ and $\langle \mathbb{P}_\alpha, \mathcal{G}_\alpha, \mathcal{F}_\alpha \mid \alpha \leq \delta \rangle$). In order to facilitate for the reader a better understanding of how symmetric iterations work, the author gives an example by using his construction in order to obtain a model in which all ultrafilters on ω (the set of natural numbers) are principal (the existence of such a model was originally established by S. Feferman [Fund. Math. **56** (1964/1965), 325–345; MR0176925]).

The next important step (Section 7 of the paper) is the justification of the term "iterated symmetric extensions", and indeed the author establishes that iterating symmetric extensions is in fact the same as doing a symmetric iteration (this is the essence of Theorem 7.8 in the paper, where it is shown that for every $\alpha < \delta$ (where δ is the length of a symmetric iteration), $\mathsf{IS}_{\alpha+1}^{G\restriction\alpha+1}$ is a symmetric extension of $\mathsf{IS}_{\alpha}^{G\restriction\alpha}$). In order to meet his goal, he first looks at two-step iterations and examines how the automorphisms and supports behave when one moves from a $\mathbb{P} * \dot{\mathbb{Q}}$-name to a $\mathbb{P}$-name for a $\dot{\mathbb{Q}}$-name, and then uses his considerations for the formulation and proof of two crucial lemmas (called 'the factorization lemmas' in the paper) of which the above main result is a corollary. In addition, the author proves that if one takes a symmetric extension of a symmetric iteration, then this could have been done by extending the original symmetric iteration by the last step (and a subtle point here is that one needs to shrink the automorphism groups—Theorem 7.9 in the paper).

The author also presents a variant of the general method where some added restrictions allow him to access filters which are not necessarily generic for the iteration.

The paper concludes with an example of a symmetric iteration in which the author subsumes the work of Monro [op. cit.] into this new framework, and shows that for all $n < \omega$ there is a model in which KWP_{n+1} holds but KWP_n fails (this is the result originally shown by Monro), and also that there is a model of ZF in which KWP_n fails for all $n < \omega$. (KWP denotes the Kinna-Wagner selection principle, which states that for every family $\mathcal{A}$ of sets, each having at least two elements, there is a function F with domain $\mathcal{A}$ such that for all $A \in \mathcal{A}$, $F(A)$ is a nonempty proper subset of A; KWP is equivalent to "Every set can be injected into the power set of an ordinal"; see [T. J. Jech, *The axiom of choice*, North-Holland, Amsterdam, 1973; MR0396271]. For $n \in \omega$, KWP_n is the statement that for every set X there exists an ordinal α such that X can be injected into $\mathcal{P}^n(\alpha)$—KWP_0 is the axiom of choice and KWP_1 is KWP.) Note that it is an open problem whether $\mathsf{KWP}_{\alpha+1} \nrightarrow \mathsf{KWP}_\alpha$ for all α (where KWP_α is the statement that every set X is equipotent with a subset of $\mathcal{P}^\alpha(\eta)$ for some ordinal η).

The author also poses some interesting open questions, one of them being the following: "Is iterating symmetric extensions the same as a single symmetric extension?" (In the context of forcing the answer is in the affirmative: iterated generic extensions can be presented as a single generic extension.)

Concluding this review, we would like to note that the paper (albeit technical) is excellently written and special care has been given to explaining all steps (and motivations) through the development of the required machinery and the new ideas that are involved. This very interesting article is certainly recommended to the reader.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.