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Berkeley cardinals and the structure of $L(V_{\delta+1})$. (English summary)
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The topic of this paper is the structural properties of the inner model $L(V_{\delta+1})$ on the assumption that δ is a singular limit of Berkeley cardinals, each of which is a limit of extendible cardinals. This assumption is in turn consistent relative to the existence of a limit club Berkeley cardinal. Berkeley cardinals and limit club Berkeley cardinals are examples of choiceless large cardinals, that is, large cardinals incompatible with the axiom of choice, and in fact the axioms postulating their existence are the strongest large cardinals axioms that have been formulated to date.

Much of the paper is concerned with studying the consequences of the theory T^*, a theory in the first-order language of set theory with one constant symbol δ, whose axioms consist of the axioms of ZF together with the assertions that δ is a limit of Berkeley cardinals which are limits of extendibles, $\mathrm{cof}(\delta) < \delta$, and $(\mathrm{cof}(\delta))^+$-DC. As discussed in the final section of concluding remarks, the main dichotomy for the future of this theory is whether it will prove to be inconsistent or whether confidence in its consistency will develop as we find that it can provide a rich mathematical structure. Prior to reading this paper, I had pretty high confidence that the first of these two would turn out to be the case. The most intriguing part of the paper to me is the suggestion in the final section that perhaps the results of this paper will instead provide some evidence that the second of these two will turn out to be the case. The existence of this dichotomy and the natural desire to resolve it is indeed one of the main motivations for studying choiceless large cardinals.

A lot of the paper is concerned with generalising the structure theory of $L(V_{\delta+1})$ in the theory $\mathrm{ZFC}+\mathrm{I}_0$, on the assumption that δ is the supremum of the critical sequence of an I_0 elementary embedding, to results about $L(V_{\delta+1})$ on the assumption that δ is a Berkeley cardinal or (a stronger assumption) a singular limit of Berkeley cardinals, each of which is a limit of extendibles. The paper provides references for the basics of the structure theory of $L(V_{\delta+1})$ in the theory $\mathrm{ZFC}+\mathrm{I}_0$. Having some knowledge of this structure theory would be a useful preliminary to reading the paper itself.

Section 1 of the paper provides some basic results about Berkeley cardinals and the inner model $L(V_{\delta+1})$, while Section 2 provides the proof of a result known as the coding lemma, which is part of the structure theory of $L(V_{\delta+1})$. Section 3 shows that the theory T^* implies that δ^+ is regular in a strong sense; there is no surjective map from V_δ onto δ^+. Section 4 shows that in the theory T^*, δ^+ is measurable in $L(V_{\delta+1})$ and also in V. Section 5 is concerned with the ordinal Θ, the least ordinal not the range of a surjective mapping in $(\mathrm{OD})^{L(V_{\delta+1})}$ with domain $V_{\delta+1}$, establishing from the theory T^* that it is a limit of measurable cardinals in $L(V_{\delta+1})$.

Readers of this paper should definitely be fully comfortable with the basics of inner models of the form $L(A)$, with forcing, and with some basic properties of extendible and supercompact cardinals. In addition, some familiarity with the structure theory of $L(V_{\delta+1})$ under $\mathrm{ZFC}+\mathrm{I}_0$ would probably be desirable. Since this paper works in a theory without the full axiom of choice, it is important to be sensitive to the issues involved when the full axiom of choice is not available. I don't think that the prerequisites required to fully understand this paper are very extensive, but I do think that it's probably fair to say that someone trying to fully understand this paper is likely to be in for a fair amount of hard work. This work has a good payoff in terms of learning more about some results on the structure theory of $L(V_{\delta+1})$ in the presence of Berkeley cardinals, which are of interest in their own right, and also might serve as a good motivator for learning the structure theory of $L(V_{\delta+1})$ under the assumption of I_0, which is connected with some fruitful research topics.

The topic of Berkeley cardinals is still relatively young, and the consistency of these cardinals should still be seen as an open problem. Thus this paper should appeal to those who would find it interesting to look at the first attempts to explore an epistemologically hazardous terrain. *Rupert McCallum*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.