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Another arithmetic of the even and the odd. (English summary)
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In [Rev. Symb. Log. **9** (2016), no. 2, 359–369; MR3506912], the reviewer asked on the one hand for a minimal axiom system of the arithmetic of the even and the odd in which one can prove the irrationality of $\sqrt{2}$, and on the other hand to prove that the irrationality of $\sqrt{17}$ cannot be proved in what was considered to be the most generous theory of the even and the odd, called $\mathcal{OE}^+_<$, which is axiomatized by the axioms for the positive cone of discretely ordered rings, called PA^-, together with two axioms: one stating that

(+) for any two numbers m and n, there are two numbers $\kappa(m,n)$ and $\mu(m,n)$, such that
$$m=\kappa(m,n)\cdot\mu(m,n) \quad \text{and} \quad n=\kappa(m,n)\cdot\mu(n,m),$$
and at least one of $\mu(m,n)$ and $\mu(n,m)$ is odd (a fact expressed by means of the unary operation symbol $[\frac{x}{2}]$);

the other axiom stating that
$$(*) \qquad \left[\frac{2x}{2}\right]=x.$$

The author of the paper under review here proves the need for the latter axiom. It must be mentioned that $(*)$ was missing both from the above-cited paper and from a later paper by S. Menn and the reviewer [Rev. Symb. Log. **9** (2016), no. 3, 638–640; MR3569173] where an error was corrected and a question answered that had been left open in [op. cit.]. By the way, the review of [MR3569173] misleadingly claims that the answer provided by Menn and the reviewer is contained in [S. T. Smith, J. Pure Appl. Algebra **79** (1992), no. 1, 63–85; MR1164123], which is not true, for the axiom system in [S. Menn and V. V. Pambuccian, op. cit.] is weaker than that of domains with a GCD.

The paper's main contribution is in providing a proper extension $\mathcal{AOE}$ of $\mathcal{OE}^+_<$ that is still a genuinely elementary theory of the even and the odd, by adding to the axioms for PA^- and to $(*)$ the following three axioms (in which $\tau(n)$ may be read as the highest power of 2 dividing n, and $\omega(n)$ as $n/\tau(n)$):

$$(1) \qquad \tau(n)=n\wedge a\cdot b=n\wedge a>1\to a=2\cdot\left[\frac{a}{2}\right],$$

$$(2) \qquad 0<n\to n=\tau(n)\cdot\omega(n)\wedge\tau(\tau(n))=\tau(n)\wedge\omega(n)=2\cdot\left[\frac{\omega(n)}{2}\right]+1,$$

$$(3) \qquad n<m\wedge\tau(m)=m\wedge\tau(n)=n\to\tau(m-n)=n.$$

The author proves that one can define in $\mathcal{AOE}$ binary operations κ and μ such that (+) holds, and that τ and ω cannot be defined in $\mathcal{OE}^+_<$, so that $\mathcal{AOE}$ is stronger than $\mathcal{OE}^+_<$. It also can be considered to be the *right* theory of the even and the odd, as Pythagoreans might have understood it. For the Pythagoreans certainly knew that a given number n cannot be indefinitely divisible by 2, and so they knew that $\tau(n)$ exists for all n. The author shows that $\mathcal{AOE}$ proves that if $\sqrt{k}$ is rational, then $\tau(k)$ is a square and $\omega(k)$ is of the form $8n+1$.

The problem regarding the impossibility of proving the irrationality of $\sqrt{17}$ in $\mathcal{A}OE$ is left open—the question originated with J. Itard's interpretation of the reason why Theodorus, as portrayed by Plato in his *Theaetetus*, stopped at $\sqrt{17}$ with his proofs of irrationality.

What we do have for the first time with this paper is the right theory in which to prove the impossibility of an irrationality proof for $\sqrt{17}$. Its solution would finally provide a rigorous definitive proof to the claim that one cannot prove by means of even and odd considerations that $\sqrt{17}$ is irrational. *Victor V. Pambuccian*

References

1. Kaye, R. (1991). *Models of Peano Arithmetic*. Oxford: Oxford University Press. MR1098499
2. Menn, S. & Pambuccian, V. (2016). Addenda et corrigenda to "The arithmetic of the even and the odd." *Review of Symbolic Logic*, **9**, 638–640. MR3569173
3. Pambuccian, V. (2016). The arithmetic of the even and the odd. *Review of Symbolic Logic*, **9**, 359–369. MR3506912
4. Pambuccian, V. (2018). A problem in Pythagorean Arithmetic. *Notre Dame Journal of Formal Logic*, **59**, 197–204. MR3778307

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.