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Burns, K. **[Burns, Keith]** (1-NW);
Climenhaga, V. **[Climenhaga, Vaughn]** (1-HST);
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Thompson, D. J. **[Thompson, Daniel J.]** (1-OHS)

Unique equilibrium states for geodesic flows in nonpositive curvature. (English summary)

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Equilibrium states (or measures) are Borel probability measure invariant under a dynamical system and defined by a thermodynamically motivated variational principle: they maximize "free energy", the maximum being called (topological) "pressure", and one of the leading special cases is a measure of maximal entropy. This notion was first studied for uniformly hyperbolic dynamical systems where their existence was among the first things established, and this was found to hold in rather great generality—for systems which are "entropy-expansive" or "h-expansive", and when R. Bowen proved this he followed his numerous examples of h-expansive systems with the "Problem. Find some differentiable maps which are *not* h-expansive" [Trans. Amer. Math. Soc. **164** (1972), 323–331; MR0285689].

Uniqueness is a much stronger requirement, and Bowen established this using the significantly more restrictive *specification property*, which is rather deeply tied in with uniform hyperbolicity. Until recently, the most notable step beyond Bowen's scope of this approach was probably the much-cited [A. B. Katok, Inst. Hautes Études Sci. Publ. Math. No. 51 (1980), 137–173; MR0573822], which focused on shadowing rather than specification.

While there was progress on equilibrium states and their uniqueness, the prospects for serious progress only brightened recently with the work of V. Climenhaga and D. J. Thompson [Adv. Math. **303** (2016), 745–799; MR3552538], which took up the Bowen approach and in an impressive feat of imagination and technical mastery developed it into a machinery ready for application in nonuniformly hyperbolic dynamics.

The present paper is one of the outstanding exemplars of such application. Its subject is the geodesic flow on a compact (rank-1) nonpositively curved Riemannian manifold. The rank-1 condition on the Riemannian metric is that there is a geodesic for which the tangent vector field is the only parallel Jacobi field, and the (open dense invariant) regular set is the set of tangent vectors to such geodesics; the singular set is its complement (and empty only in the well-understood uniformly hyperbolic case).

This kind of geodesic flow is the original exemplar which motivated the development of the theory of nonuniform hyperbolicity because here one obtains (in a nonuniform way) complete hyperbolicity. The precursor to this paper is the celebrated [G. Knieper, Ann. of Math. (2) **148** (1998), no. 1, 291–314; MR1652924], which produced uniqueness of the measure of maximal entropy for this situation—with quite different tools. The elapse of two decades underscores the difficulty of the subject and the need for completely new methods.

Unlike with entropy, uniqueness of equilibrium states is also connected to properties of the "potential" function that defines the free energy the measure is to maximize. Accordingly, this paper produces uniqueness results under various assumptions on this potential function. Specifically, the authors give conditions under which their new techniques can be applied to geodesic flows on rank-1 manifolds and demonstrate that these conditions are satisfied for large classes of potential functions. They show (Theorem A, obtained from the main result, Theorem 3.1) that a sufficient condition on (sufficiently regular) potential functions is the presence of a *pressure gap*: the singular set does not carry full pressure, meaning that the restriction of the geodesic flow to it has smaller pressure than on the complement or, equivalently, in total.

As applications, they obtain that for geodesic flows on surfaces, uniqueness holds for scalar multiples of the geometric potential if the scalar is in the interval $(-\infty, 1)$, which is optimal (Theorem C); indeed, the pressure varies C^1 with the scalar factor, and the equilibrium state is hyperbolic, fully supported, Bernoulli, and the weak*-limit of weighted regular closed geodesics. In higher dimensions, they obtain the same result for scale factors in a neighborhood of 0 (Theorem D), and give examples where uniqueness holds on all of $\mathbb{R}$. The geometric potential is the Jacobian of the flow in the expanding direction, and this, which leads to the Sinai-Ruelle-Bowen measure, is the other leading case of equilibrium states after the measure of maximal entropy, but with the serious additional challenge that this potential function is usually less regular than needed for the application of any standard theory. Accordingly, the aforementioned Theorem A applies to a potential that is either Hölder continuous or a constant multiple of the geometric potential.

They also give criteria that imply the required pressure gap. The pressure gap occurs, for instance, whenever the potential is locally constant on a neighborhood of the singular set (Theorem B), which allows them to give examples for which uniqueness holds on a C^0-open and dense set of Hölder potentials.

It is easy to miss an insight provided here related to the *entropy* gap. It is a corollary of Knieper's uniqueness proof for the measure of maximal entropy [op. cit.] that in the present context the singular set carries less topological entropy than the regular set—but this does not produce a direct constructive proof of the entropy gap. In establishing the needed pressure gap (§8), the present paper produces such an argument: approximate singular orbit segments by regular orbit segments having the specification property, and use these to build a collection of orbits with greater topological entropy/pressure than the singular set. In particular, applied with the potential equal to zero, this work reproves the result by Knieper using dynamically direct methods. *Boris Hasselblatt*

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