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Bartholdi, Laurent (F-ENS-NDM); **Dudko, Dzmitry** (F-ENS-NDM)
Algorithmic aspects of branched coverings IV/V. Expanding maps. (English summary)
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This is the fourth in a series of papers by the authors. An overview of the series was given in [L. Bartholdi and D. Dudko, Ann. Fac. Sci. Toulouse Math. (6) **26** (2017), no. 5, 1219–1296; MR3746628].

The next two paragraphs give some context and motivation, from the perspective of this reviewer.

Fix a closed oriented surface $S = S_{g,n}$ of genus g equipped with a finite set $P \subset S$ of n distinguished points. The *mapping class group* $\mathrm{Mod}(S)$ is now well studied. The *Nielsen-Thurston classification* implies a geometrization result. For a precise formulation, see [B. Farb and D. Margalit, *A primer on mapping class groups*, Princeton Math. Ser., 49, Princeton Univ. Press, Princeton, NJ, 2012 (Theorem 13.2 and Corollary 13.3); MR2850125]. Roughly, this geometrization result asserts that given any $f \in \mathrm{Mod}(S)$, there is a canonical (possibly empty) minimal multicurve $\{c_1, \dots, c_m\}$ invariant by f such that some power f^k sends each complementary component to itself, and is *geometrizable*: it is either isotopic to the identity map, or to a so-called *pseudo-Anosov* map. It also expresses the restriction of f^k to an annular neighborhood of one of the c_i as a power of a Dehn twist. A pseudo-Anosov map is, by definition, affine in suitable coordinates. Furthermore, there are practical algorithms to find this canonical decomposition, given an expression of f as a word in the generators; see, for example, the recent polynomial-time announcement by M. C. Bell and R. C. H. Webb ["Polynomial-time algorithms for the curve graph", preprint, arXiv:1609.09392.pdf].

The series of papers by the authors, in this reviewer's opinion, may be fruitfully summarized as follows. Imagine now that $f\colon (S, P) \to (S, P)$ is no longer a homeomorphism, but an orientation-preserving *branched covering map* of some degree $d \geq 2$ for which the set of branch values lies in P. The Riemann-Hurwitz formula implies that either $g = 1$ and f is a covering, or $g = 0$. In the latter case, f is called a *Thurston map*. W. Thurston introduced such maps as combinatorial objects arising naturally in the classification theory of rational maps acting as dynamical systems on the Riemann sphere. Just like surface homeomorphisms, Thurston maps $f\colon (S^2, P) \to (S^2, P)$ with branch values in P may be composed, and their isotopy classes relative to P thus form a countable semigroup. The analog of conjugacy in this category is called *combinatorial equivalence*, which one can read as conjugacy-up-to-isotopy. The series of papers by the authors develops, in a systematic way, a parallel theory to that which currently exists for mapping class groups. Their development requires some significant innovations. They use recently introduced algebraic objects, called *bisets*, to encode the homotopy class of a Thurston map; one may view this as a generalization of the classical Baer-Dehn-Nielsen theorem encoding mapping classes as group automorphisms. To deal with decomposing Thurston maps, they need a version of van Kampen's theorem; this is the substance of Parts I and II of the series [L. Bartholdi and D. Dudko, Groups Geom. Dyn. **12** (2018), no. 1, 121–172; MR3781419; "Algorithmic aspects of branched coverings II/V. Sphere bisets and their decompositions", preprint, arXiv:1603.04059]. The set of homotopy classes of Thurston maps forms a biset over the mapping class group.

Its algebraic structure is extremely rich, and this structure features prominently in the authors' analysis.

The paper under review gives a characterization of similarly geometrizable Thurston maps. Here, 'geometrizable' is interpreted as admitting a so-called *Böttcher expanding* representative. A Thurston map is Böttcher expanding if (i) it is smooth; (ii) there is a complete orbifold Riemannian metric on the complement of the forward orbits of periodic branch points which is expanded by f; and (iii) the first-return map in a neighborhood of a periodic branch point is locally conjugate to $z \mapsto z^k$ near the origin, $k \geq 2$. Rational Thurston maps give a wealth of examples. Given a homotopy class of a Thurston map, a Böttcher expanding representative, if it exists, is unique up to topological conjugacy (Corollary 1.2); this is an analog of the result that an irreducible mapping class has, up to topological conjugacy, a unique pseudo-Anosov representative. Following is the main result of the paper:

Theorem A. Let $f\colon (S^2, P) \to (S^2, P)$ be a Thurston map, not doubly covered by a torus endomorphism. The following are equivalent:

(1) f is combinatorially equivalent to a Böttcher expanding map;
(2) f is combinatorially equivalent to a topologically expanding map;
(3) the biset $B(f)$ is an orbisphere contracting biset;
(4) f is noninvertible and admits no Levy cycle.

A *Levy cycle* is a multicurve whose elements are permuted, with each element mapping by a homeomorphism. A major consequence of Theorem A is the authors' Corollary B. This may be fruitfully viewed as a first step in an analog of the aforementioned geometrization of mapping classes. It asserts that each Thurston map has a canonical *Levy obstruction* consisting of a canonical Levy cycle, and a corresponding decomposition. Cutting along the canonical Levy obstruction and taking first-return maps yields "smaller" Thurston maps that are either Böttcher expanding, or surface homeomorphisms. (The surface homeomorphisms could, if one wished, be then further decomposed into geometrizable pieces—but this is not the authors' focus.)

The study of Thurston maps doubly covered by torus endomorphisms reduces to analyzing unbranched maps on tori; the exclusion of such maps in the hypothesis of Theorem A is typical in this field. In Theorem A, condition (3) is algebraic, while condition (4) is a combinatorial-dynamical condition, which is equivalent to the canonical Levy cycle being empty.

In the proof of Theorem A, the difficult implication is (4) $\Rightarrow$ (1). The proof is along the following lines. The authors first apply the *rational canonical decomposition* of f, shown to exist by work of this reviewer [Adv. Math. **158** (2001), no. 2, 154–168; MR1822682] and enhanced by N. Selinger [Invent. Math. **189** (2012), no. 1, 111–142; MR2929084]. This involves cutting along another canonical multicurve comprising obstructions to equivalence to a rational map. This canonical multicurve may be strictly larger than the canonical Levy multicurve in Corollary B. Condition (4) implies that the "pieces" in the canonical rational decomposition are equivalent to rational maps, and are therefore Böttcher expanding. If the canonical Levy obstruction is empty, but its rational canonical obstruction is nonempty, the authors show by natural but somewhat delicate estimates how to glue together the expanded metrics on the pieces to obtain a Böttcher expanding representative.

The authors give applications to numerous natural questions regarding decidability and realizability of certain types of combinations, e.g. matings.

{For Part III see [L. Bartholdi and D. Dudko, "Algorithmic aspects of branched coverings III/V. Erasing maps, orbispaces, and the Birman exact sequence", preprint, arXiv:1802.03045].}

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.