MR3834666 37C85 37C35 37C99
Ponce, Gabriel (BR-ECPM)
Generating positive geometric entropy from recurrent leaves. (English summary)
Proc. Amer. Math. Soc. **146** (2018), *no. 10,* 4389–4404.

The author gives a sufficient condition for a codimension one foliation on a closed 3-manifold to be approximated in the C^r topology ($r \geq 1$) by another foliation with positive entropy in the sense of [É. Ghys, R. Langevin and P. G. Walczak, Acta Math. **160** (1988), no. 1-2, 105–142; MR0926526]; this result is also applied to exhibit a Riemannian foliation that can be approximated by foliations with positive entropy.

The paper is difficult to read due to several typos and minor errors; moreover, some of the intermediate lemmas seem to be rather classical in the theory of foliations, and the author offers, in the opinion of this reviewer, unnecessarily complicated proofs. The paper is aimed at those researchers interested in foliation theory and 1-dimensional dynamics. The main theorem (Theorem A) can be written in the following equivalent form:

Theorem A: Let $\mathcal{F}$ be a C^r, $r \geq 1$, codimension 1 foliation on a closed 3-manifold. Suppose that there exists a recurrent leaf $L \in \mathcal{F}$ with two homotopically non-trivial loops γ, σ with non-trivial homological intersection so that their homotopy classes have infinite order. Assume that σ has trivial holonomy; then, for all $\varepsilon > 0$, there exists a foliation $\mathcal{G}_\varepsilon$ with positive geometric entropy and ε close to $\mathcal{F}$ in the C^r topology.

In order to prove Theorem A the author uses the fact that C^r codimension one foliations with positive entropy are precisely those that have a resilient leaf, i.e., a recurrent leaf with a loop whose holonomy is contractive. This equivalence was proved for class C^r, $r \geq 2$, in [op. cit.], and improved to class C^1 by Walczak [*Dynamics of foliations, groups and pseudogroups*, IMPAN Monogr. Mat. (N. S.), 64, Birkhäuser, Basel, 2004; MR2056374]. The author points out that the result does not follow in the C^∞ topology (Remark 4.3), but it seems that he is not considering the usual Whitney topology, so, as far as I could check, the result seems to be also true in that regularity.

The idea of the proof of Theorem A is straightforward: make a local perturbation in the transverse direction (in L) to γ so that the holonomy of γ becomes contractive. The perturbation is made so that both loops in L survive to the perturbation, i.e., they are still tangent to some leaf L' of the perturbed foliation. Since L is assumed to be recurrent, the perturbation can be performed in such a way that L' must still meet the basin of attraction of the perturbed holonomy map associated to γ. Thus, L' becomes resilient and therefore the geometric entropy of the perturbed foliation must be positive.

Corollary B says that there exists a Riemannian foliation (thus with zero entropy) that can be approximated by leaves with positive entropy. Take any minimal Riemannian foliation where all leaves have infinite genus; then Theorem A can be applied to any leaf.

The first and second sections are devoted to introducing the main results and the basics on foliations and geometric entropy. The third and fourth sections deal with the method of local perturbation used to prove the main theorem; for this purpose two concepts are introduced: the *family of jointly splitting charts* and the *water sliding diffeomorphisms*.

The jointly splitting charts are just foliated charts that must cover a product neighborhood of σ in a suitable way. Its existence is provided by the classical Reeb stability theorem applied to compact domains. The author doesn't point out this observation, which makes the statement of Theorem A in the paper unnecessarily complicated.

The water slide diffeomorphism is the tool used by the author to formalize the following idea: let $\Sigma \equiv \sigma \times (-\rho, \rho)$ be a tubular neighborhood of σ in L; by Reeb stability there exists a neighborhood of Σ in the ambient manifold that is diffeomorphic to a product foliation $\Sigma \times (-\delta, \delta)$ where $\Sigma \equiv \Sigma \times \{0\}$. Choose a C^r diffeomorphism $f\colon (-\delta, \delta) \to (-\delta, \delta)$ that is C^r tangent to the identity on the extreme points and fixes 0 (this is the transverse coordinate of the water sliding map). Then consider the perturbed foliation $\Sigma \times_f (-\delta, \delta)$ given also by cylinders but gluing $\sigma \times \{-\rho\} \times \{t\}$ with $\sigma \times \{\rho\} \times \{f(t)\}$; this foliation can be easily smoothed in its transverse boundary so that, when changing $\Sigma \times (-\delta, \delta)$ by $\Sigma \times_f (-\delta, \delta)$, the resulting foliation is still C^r. If δ is sufficiently small and since 0 is a fixed point of f, the loops σ and γ will survive to the perturbation as tangent loops. It is classical in 1-dimensional dynamics how to transform the holonomy map associated to γ (or γ^{-1}) into a contractive one by a perturbation arbitrarily close to the identity (recall that this can be done sufficiently close to the base point).

Corollary B is not new in the theory of foliations; it is straightforward to choose two hyperbolic diffeomorphisms of the circle that are close to the identity and so that some iterates of them play the dynamical Ping-Pong game. Assuming regularity C^2 or higher, the Sacksteder theorem guarantees the existence of a hyperbolic fixed point in the minimal set. Thus the suspension of this pair of diffeomorphisms over the bitorus is a foliation with positive entropy and it is C^r close to the product foliation (which is, of course, Riemannian).

Section five is devoted to the proof of Theorem A and the sixth and last section offers some comments and problems. In Problem 1, the author asks for examples of foliations that have zero entropy and that cannot be approximated by positive entropy ones (i.e., robustly zero entropy). There are lots of such examples: the simplest ones are manifolds foliated by compact simply connected leaves; such foliations are rigid by Reeb stability. Another family of examples are the compact-Hausdorff foliations where the generic leaf has trivial first homology group; these are also rigid foliations by [D. B. A. Epstein and H. Rosenberg, in *Geometry and topology*, 151–160, Lecture Notes in Math., 597, Springer, Berlin, 1977; MR0501007]. Another example is any suspension over the torus: any small perturbation of this kind of foliation is still a suspension and therefore its leaves have polynomial growth (since they are covered by the euclidean plane); therefore it cannot have resilient leaves. Problem 2 is more interesting, asking for a classification of robustly zero entropy foliations; this seems a very interesting question open to new results and examples.

Carlos Meniño Cotón

References

1. Rufus Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. **153** (1971), 401–414. MR0274707 MR0274707
2. Keith Burns and Howard Weiss, *A geometric criterion for positive topological entropy*, Comm. Math. Phys. **172** (1995), no. 1, 95–118. MR1346373 MR1346373
3. César Camacho and Alcides Lins Neto, *Geometric theory of foliations*, translated from the Portuguese by Sue E. Goodman, Birkhäuser Boston, Inc., Boston, MA, 1985. MR824240 MR0824240
4. Alberto Candel and Lawrence Conlon, *Foliations. I*, Graduate Studies in Mathematics, vol. 23, American Mathematical Society, Providence, RI, 2000. MR1732868 MR1732868
5. Sylvain Crovisier, *Birth of homoclinic intersections: a model for the central dynamics*

of partially hyperbolic systems (English, with English and French summaries), Ann. of Math. (2) **172** (2010), no. 3, 1641–1677. MR2726096 MR2726096

6. D. B. A. Epstein, *A topology for the space of foliations*, Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), Lecture Notes in Math., Vol. 597, Springer, Berlin, 1977, pp. 132–150. MR0501006 MR0501006
7. D. B. A. Epstein, K. C. Millett, and D. Tischler, *Leaves without holonomy*, J. London Math. Soc. (2) **16** (1977), no. 3, 548–552. MR0464259 MR0464259
8. É. Ghys, R. Langevin, and P. Walczak, *Entropie géométrique des feuilletages* (French), Acta Math. **160** (1988), no. 1–2, 105–142. MR926526 MR0926526
9. Gilbert Hector, *Feuilletages en cylindres* (French), Geometry and topology (Proc. III Latin Amer. School of Math., Inst. Mat. Pura Aplicada CNPq, Rio de Janeiro, 1976), Lecture Notes in Math., Vol. 597, Springer, Berlin, 1977, pp. 252–270. MR0451260 MR0451260
10. S. Hurder, *Entropy and dynamics of C^1 foliations*, preprint, available from http://homepages.math.uic.edu/~hurder/publications.html, 2000.
11. Steven Hurder, *Dynamics and the Godbillon-Vey class: a history and survey*, Foliations: geometry and dynamics (Warsaw, 2000), World Sci. Publ., River Edge, NJ, 2002, pp. 29–60. MR1882764 MR1882764
12. Steven Hurder, *Classifying foliations*, Foliations, geometry, and topology, Contemp. Math., vol. 498, Amer. Math. Soc., Providence, RI, 2009, pp. 1–65. MR2664589 MR2664589
13. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817. MR0228014 MR0228014

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.