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★**Cantor minimal systems.**
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The book under review is an introductory course in the orbit equivalence theory of Cantor minimal systems. The main goal of the book is to explore when two dynamical systems on a Cantor set have the same orbit structure. The author shows the beautiful interplay between Cantor minimal systems, Bratteli diagrams and ordered abelian groups. The book is a wonderful introduction to the topic for students who have a good undergraduate mathematics background and want to begin pursuing research. The chapters are written in a clear and compelling way, and the book contains numerous exercises, references and historical remarks.

Throughout the review, by a *Cantor system* we will mean a pair which consists of a Cantor space X and a homeomorphism φ from X onto itself.

The first chapter of the book concentrates on the dyadic odometer $(\{0,1\}^{\mathbb{N}},\varphi)$ as an example of a Cantor minimal system. The main focus of the book is illustrated by showing the connections between the orbit relation R_φ generated by φ and the tail equivalence relation R on $\{0,1\}^{\mathbb{N}}$. The second chapter provides a background on Cantor sets and orbit equivalence relations. In particular, the author discusses orbit relations for group actions and minimal equivalence relations.

In Chapters 3, 4 and 5, connections between Cantor minimal systems and Bratteli diagrams are considered. In particular, Chapters 3 and 4 introduce Bratteli diagrams as generalizations of dyadic odometers. In Chapter 3, a Bratteli diagram (V,E) and its space of infinite paths X_E are defined. The corresponding AF-equivalence relation R_E (also called tail equivalence relation) is introduced, the operation of telescoping and the notion of a simple Bratteli diagram are discussed. In Chapter 4, an ordered Bratteli diagram $(V,E,\geq)$ and a Vershik map φ_E are defined, the relation between R_E and R_{φ_E} is considered. In Chapter 5, the proof of the fact that every minimal homeomorphism of a Cantor set is topologically conjugate to a Vershik map on a simple properly ordered Bratteli diagram is outlined.

Chapter 6 generalizes the notions of the orbit relation and the AF-relation by introducing étale equivalence relations on a topological space. These are equivalence relations together with a topology, which arises from a local action of a collection of partial homeomorphisms. The characterization of the minimality of an étale equivalence relation on a Cantor set is given. It is proved that, for a Bratteli diagram, the corresponding AF-equivalence relation R_E is étale. For a free action φ of a group G on a Cantor set, the orbit relation R_φ is also an étale equivalence relation. It is shown that relations R_E and R_φ for a minimal free action φ of $\mathbb{Z}$ on a Cantor set are not isomorphic, since R_φ is compactly generated and R_E is not.

Chapter 7 is devoted to the introduction of the so-called D invariant $D(X,R)$ for étale equivalence relations. Namely, Section 1 deals with the countable abelian group $C(X,\mathbb{Z})$ of all continuous integer-valued functions on a Cantor set X, with the group operation of addition. Section 2 introduces ordered abelian groups. In Section 3, the D invariant for an étale equivalence relation R is defined as a certain quotient group $C(X,\mathbb{Z})/B(X,\mathbb{Z})$, where $B(X,\mathbb{Z})$ is a subgroup defined with the help of partial homeomorphisms generating the topology on R. It is shown that isomorphic étale equivalence relations have isomorphic D invariants. In Section 4, the author constructs inductive limits of abelian groups. Section 5 is devoted to constructing the dimension group of a Bratteli diagram. The author turns vertex sets of a Bratteli diagram into abelian groups and edge sets into positive group homomorphisms, creating the dimension group as an inductive limit. For practical purposes, one can view the group generated by a vertex set V_n with k_n vertices as a group of column vectors $\mathbb{Z}^{k_n}$, and the corresponding homomorphism h_{E_n} as a $k_n \times k_{n-1}$ matrix (such matrices are also called the incidence matrices of a diagram). A few examples are considered; a characterization of a simple Bratteli diagram in terms of dimension groups is given. In Section 6, it is proved that the D invariant $D(X_E, R_E)$ for an AF-equivalence relation R_E is isomorphic as an ordered abelian group with order distinguished unit to the dimension group of the corresponding Bratteli diagram. In Section 7, the D invariant $D(X_E, R_{\varphi_E})$ is computed for the orbit relation R_{φ_E} generated by a Vershik map φ_E on a properly ordered Bratteli diagram. It is shown that for a properly ordered Bratteli diagram, the D invariants $D(X_E, R_{\varphi_E})$ and $D(X_E, R_E)$ are isomorphic as ordered abelian groups with order distinguished units.

Chapter 8 is devoted to the Effros-Handelman-Shen theorem, which provides a criterion for an ordered abelian group to be a dimension group. In other words, the theorem shows when an ordered abelian group is constructed from a Bratteli diagram. The chapter contains a detailed proof. This beautiful result shows a way of producing a Bratteli diagram from an ordered abelian group, but it is difficult to use in practice.

In Chapter 9, the Bratteli-Elliott-Krieger theorem is presented. It shows that for two AF-equivalence relations, the D invariant is a complete invariant of isomorphisms. This theorem is the first deep result in the book towards the goal of understanding orbit equivalence.

Chapter 10 is devoted to strong orbit equivalence. Two free minimal actions of $\mathbb{Z}$ on Cantor sets are strong orbit equivalent if there exists an orbit equivalence such that the corresponding orbit cocycles each have at most one point of discontinuity. It is proved that two minimal actions of $\mathbb{Z}$ are strong orbit equivalent if and only if the corresponding D invariants are isomorphic as ordered abelian groups with order unit. This result was one of the motivations to introduce the notion of strong orbit equivalence.

Chapter 11 introduces an invariant of orbit equivalence for étale equivalence relations. The chapter consists of eight sections. Section 1 is an introduction to the measure theory for those readers who have no background in the area. It focuses on the integration of continuous integer-valued functions on a compact totally disconnected metric space X. Section 2 is devoted to the states on ordered abelian groups and explores connections between probability measures on X and states on $C(X,\mathbb{Z})$. Section 3 introduces the notion of a probability R-invariant measure for an étale equivalence relation R on a compact, totally disconnected metric space X. Section 4 shows the correspondence between probability R-invariant measures on X and states on the D invariant $D(X,R)$. Section 5 introduces an invariant for étale equivalence relations, which is denoted by $D_m(X,R)$. It is an ordered abelian group which can be computed from $D(X,R)$. It is proved that D_m is an invariant of orbit equivalence of étale equivalence relations. Section 6 shows how to compute D_m invariant for an AF-equivalence relation R_E. In Section 7, it is proved that étale equivalence relations R_E and R_{φ_E} on a properly ordered Bratteli diagram, where φ_E acts freely, have the same invariant measures. As a consequence, it follows that for such a Bratteli diagram, the groups $D_m(X_E, R_E)$ and $D_m(X_E, R_{\varphi_E})$ are isomorphic as ordered abelian groups with order units. Section 8 is devoted to the classification of odometers. The author shows how to define an odometer as a Bratteli-Vershik system. It is proved that, for odometers, the D and D_m invariants are isomorphic, and a formula for computing these invariants is given. Moreover, it is shown that two odometers are topologically conjugate if and only if they are orbit equivalent.

Chapter 12 is devoted to the study of orbit equivalence for minimal AF-equivalence relations on an infinite space X. The author states and proves a simple version of the absorption theorem which tells that if R is a minimal AF-equivalence relation on X and $x, \overline{x}$ are two arbitrary points in X, then (X, R) is orbit equivalent to $(X, \tilde{R})$, where $\tilde{R}$ is the smallest equivalence relation containing R and $(x, \overline{x})$. This means that there exists a homeomorphism h of X such that $h \times h(\tilde{R}) = R$ and $\tilde{R}$ is in fact an étale equivalence relation. It also means that a minimal AF-equivalence relation can "absorb a small extension". Matui's absorption theorem, a more general and technical result, is stated at the end of the chapter without a proof.

Chapter 13 states the classification theorem for minimal AF-equivalence relations up to orbit equivalence. Two minimal AF-equivalence relations are orbit equivalent if and only if their D_m invariants are isomorphic as ordered abelian groups with order units. An example is given which illustrates the main ideas and shows that the notions of isomorphism and orbit equivalence are different.

In Chapter 14, the classification of AF-equivalence relations up to orbit equivalence is extended to include minimal $\mathbb{Z}$-actions. It is proved that two equivalence relations on Cantor sets, such that each of them is either a minimal AF-equivalence relation or the orbit relation of a minimal action of $\mathbb{Z}$, are orbit equivalent if and only if the corresponding D_m invariants are isomorphic as ordered abelian groups with order units. It is noted that this classification result holds for minimal actions of finitely generated free abelian groups.

An appendix contains Bratteli-Vershik models for rotations of a disconnected circle and substitution dynamical systems.

As said before, the book is a very nicely written introduction to the orbit equivalence theory of Cantor minimal systems. I highly recommend it for students who would like to conduct research in this area. There are numerous exercises which allow one to learn the introduced concepts through experience. Many results are mentioned without proofs, since these are either straightforward or left as an exercise for the reader. The main aim of the author is to develop the theory to the point when the statements of the classification results may be understood and appreciated by the reader, while the proofs are presented for special cases which give the main idea but do not contain many technical difficulties. The book gives the intuition needed to work in this area and the inspiration for further research. The style of writing is very encouraging and leaves a reader with an impression of being at the lecture and listening to the author.

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