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★**Foundations of rigid geometry. I.**

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The original aim of rigid analytic geometry was to develop a workable analogue of complex analytic geometry over non-archimedean fields, such as $\mathbb{Q}_p$ or its completed algebraic closure $\mathbb{C}_p$. The adjective 'rigid' comes from J. T. Tate's pioneering approach [Invent. Math. **12** (1971), 257–289; MR0306196] to circumventing the problems caused by the total disconnectedness of non-archimedean topologies, by imposing 'rigidity' conditions on coverings in order to ensure good local-to-global properties of analytic functions.

Since Tate's original work, other approaches to non-archimedean analytic geometry have been proposed, two principal examples being V. G. Berkovich's theory of analytic spaces [*Spectral theory and analytic geometry over non-Archimedean fields*, Math. Surveys Monogr., 33, Amer. Math. Soc., Providence, RI, 1990; MR1070709] and R. Huber's theory of adic spaces [*Étale cohomology of rigid analytic varieties and adic spaces*, Aspects Math., E30, Friedr. Vieweg, Braunschweig, 1996; MR1734903]. An earlier alternative was suggested by M. Raynaud [in *Table Ronde d'Analyse non archimédienne (Paris, 1972)*, 319–327, Bull. Soc. Math. France, Mém. 39–40, Soc. Math. France, Paris, 1974; MR0470254] and takes as its point of departure the following theorem: If K is a non-archimedean field and $\mathcal{V}$ its ring of integers, then the generic fibre functor

$$\mathfrak{X} \mapsto \mathfrak{X}_K$$

from quasi-compact admissible formal schemes over $\mathcal{V}$ to quasi-compact quasi-separated rigid analytic spaces over K becomes an equivalence after inverting 'admissible blowups' of formal schemes (that is, blowups of coherent, open ideals); for details see [S. Bosch and W. Lütkebohmert, Math. Ann. **295** (1993), no. 2, 291–317; MR1202394]. The point is that this result can then be turned on its head and be used to *define* rigid analytic spaces. In this way, formal geometry becomes a *model* for rigid geometry, a *model* for a given rigid space $\mathcal{X}$ being a formal scheme $\mathfrak{X}$ such that $\mathfrak{X}_K \cong \mathcal{X}$. The power of this approach is that one can reduce questions in rigid geometry to similar questions in formal geometry by the process of taking models, as was done in the series of papers [S. Bosch and W. Lütkebohmert, op. cit.; Math. Ann. **296** (1993), no. 3, 403–429; MR1225983; S. Bosch, W. Lütkebohmert and M. Raynaud, Math. Ann. **302** (1995), no. 1, 1–29; MR1329445; Invent. Math. **119** (1995), no. 2, 361–398; MR1312505].

If one is interested mainly in analytic varieties locally of finite type over a fixed non-archimedean field, then all these approaches are essentially equivalent (in the sense that they give rise to the same category of rigid analytic spaces, at least under mild finiteness hypotheses). However, the approaches of Berkovich and Huber have the advantage of allowing for a broader class of analytic spaces than those appearing in the theories of Tate or Raynaud. Huber's theory in particular has an extremely wide scope, taking as its affine building blocks pairs $(A^{\triangleright}, A^{+})$ consisting of a topological ring $A^{\triangleright}$ admitting an open adic subring with finitely generated ideal of definition, and an open, integrally closed subring A^{+}.

The aim of the book under review is to develop Raynaud's viewpoint of analytic geometry as a localisation of formal geometry in the greatest generality possible, roughly comparable with that provided by Huber's theory of adic spaces. The key definition is essentially the same as that given by Raynaud: the category of coherent rigid spaces is the localisation of the category of coherent, adic formal schemes of finite ideal type along the class of admissible blowups; more general rigid spaces are then obtained by gluing. The advance represented by this book lies in vastly enlarging the kind of formal schemes that are permissible as 'models' for rigid analytic spaces.

A significant part of the difficulty in carrying through this program successfully comes from the need to work systematically with non-Noetherian formal schemes, even if one is ultimately interested in studying Noetherian rigid spaces. In fact, this difficulty can already be seen in the simplest possible case, namely, the point $\mathrm{Sp}(\mathbb{C}_p)$. This is certainly a Noetherian rigid space; however, its natural formal model $\mathrm{Spf}(\mathcal{O}_{\mathbb{C}_p})$ is certainly not Noetherian as a formal scheme.

In order to overcome these difficulties, the authors make use of two finiteness conditions on adic formal schemes of finite ideal type, namely 'universally rigid-Noetherian' and the stronger condition 'universally adhesive', which they introduce. For an affine formal scheme $\mathrm{Spf}(A)$, with finitely generated ideal of definition $I \lhd A$, being universally adhesive amounts to requiring that the scheme $\mathrm{Spec}(A) \setminus V(I)$ is Noetherian, that finitely generated A-modules have finitely generated I-torsion, and that both remain true for any topologically finite-type A-algebra. Universally rigid-Noetherian is similar, but without the condition on finite generation of I-torsion. (Note that the 'universally Noetherian' condition also plays a rôle in Huber's theory of adic spaces, since it is one of the conditions that guarantees that the structural presheaf $\mathcal{O}_{\mathcal{X}}$ of an adic space is in fact a sheaf.) They then go on to systematically (re-)develop the theory of formal schemes, taking as much care as possible only to impose finiteness conditions when necessary.

This partly accounts for the length of the book (the definition of a coherent rigid space referred to above is given on page 472), since the authors must re-prove many foundational results in formal geometry, but without Noetherian hypotheses. Another factor is the authors' stated aim to make the work as self-contained as possible. The extent to which this latter aim is achieved is impressive; however, as should probably be expected, the book's self-containment is perhaps more in the style of an encyclopaedic reference on the subject, and less as a place to first learn about formal and rigid geometry.

There is significant overlap between the approach taken in this book and that of A. Abbes in [*Éléments de géométrie rigide. Volume I*, Progr. Math., 286, Birkhäuser/Springer Basel AG, Basel, 2010; MR2815110], where a theory of rigid spaces is developed by applying Raynaud's method to the category of 'idyllic' formal schemes. These are formal schemes locally modelled on topological rings that are either Noetherian or topologically of finite presentation over separated and complete height 1 valuation rings. No specific comparison between idyllic and universally adhesive formal schemes is stated in this book; however, rings topologically of finite type over separated and complete valuation rings appear as 'topological algebras of type (V)', and these are shown to be (topologically) universally adhesive.

The current book is intended to be the first volume of a much larger project, where more advanced topics in the theory of rigid spaces will be discussed. In the next volume, the authors intend to cover the formal flattening theorem on the existence of flat models, the equivalence of the 'three definitions of properness', and an 'Equivalence Theorem' showing that no extra generality is obtained by allowing formal algebraic spaces (rather than just formal schemes) as models for rigid spaces.

Let me now give a brief overview of the contents of the book.

Part 0 mostly covers background material that is needed in the rest of the book, on a wide range of topics including general topology, homological algebra, algebraic spaces and valuation rings. There are, however, some new concepts introduced here, for example, the notion of a valuative topological space as a locally coherent and sober topological space such that the set of generalisations G_x of any given point x is totally ordered. This captures the good properties enjoyed by Huber's adic spaces, or equivalently by the 'visualisation' of the authors' rigid spaces. Other important definitions appearing here are that of 'topologically universally adhesive' and 'topologically universally rigid-Noetherian' for pairs (A, I) that were sketched above. The notion of a 'topological algebra of type (V)' also mentioned above is introduced; these provide the affine building blocks for formal models of 'classical' rigid analytic spaces, that is, those locally of finite type over non-archimedean fields.

Part I is concerned with a systematic development of the theory of formal geometry, avoiding Noetherian hypotheses as much as possible. The basic class of formal schemes considered is adic formal schemes of finite ideal type, and the finiteness conditions 'universally adhesive' and 'universally rigid-Noetherian' are globalised. The authors study 'adically quasi-coherent sheaves' on formal schemes, which are globalised versions of I-adically complete modules over an I-adically topologised ring A, and also discuss various standard properties of morphisms of formal schemes, such as flatness, properness, smoothness, etc. A theory of formal algebraic spaces is developed, again, crucially, with much weaker finiteness hypotheses than has been done previously. Finally, they study the cohomology of adically quasi-coherent sheaves and prove finiteness and GFGA theorems. The weaker Noetherian-like conditions defined earlier play an important rôle here, with the relevant GFGA theorems being proved initially for universally adhesive formal schemes, before being extended to the universally rigid-Noetherian case.

Part II is the real heart of the book. The definition of a rigid space is given, and various foundational aspects of the theory of these rigid spaces are studied. Standard properties of morphisms between rigid spaces (open immersions, closed immersions, separated morphisms, etc.) are defined, and they are related to the corresponding properties of morphisms between formal models. Coherent sheaves and their cohomology are discussed and the notion of a 'Stein' affinoid introduced (note that if an affine formal scheme does not admit a locally principal ideal of definition, its generic fibre may not satisfy Theorems A and B). The visualisation $\langle \mathcal{X} \rangle$ of a rigid space $\mathcal{X}$ is constructed as the projective limit of the underlying topological spaces of all models of $\mathcal{X}$, and exactly as in Huber's theory of adic spaces this is equipped with a 'rational' structure sheaf and an 'integral' structure sheaf. Other topics covered include a discussion of points in rigid geometry, the GAGA functor and comparison theorem, dimension theory, and the maximum modulus principle. There are also several appendices, which, among other things, prove important comparison theorems between the rigid spaces introduced here and those defined by Tate, Berkovich and Huber respectively. For example, the authors show that if $\mathcal{X}$ is a locally universally Noetherian rigid space, then its visualisation $\langle \mathcal{X} \rangle$, equipped with its rational and integral structure sheaves, is an adic space in the sense of Huber, and the visualisation functor induces an equivalence of categories between rigid spaces locally of finite type over $\mathcal{X}$ and adic spaces locally of finite type over $\langle \mathcal{X} \rangle$. They also show that the functor $\mathcal{X} \mapsto \langle \mathcal{X} \rangle$ from locally universally Noetherian rigid spaces to adic spaces is faithful, but it is not clear whether or not they expect it to be full in general. Another key comparison result states that the maximal separated quotient $[\mathcal{X}]$ of the visualisation $\langle \mathcal{X} \rangle$ can be identified with the topological space underlying the Berkovich space associated to $\mathcal{X}$. Similarities between the approach to rigid spaces taken here and the classical theory of Zariski-Riemann spaces are also discussed.

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