MR1118699 (92j:35050) 35J60 35B05 35D05 35G20
Crandall, Michael G. (1-UCSB); **Ishii, Hitoshi** (J-CHUO);
Lions, Pierre-Louis (F-PARIS9-A)

User's guide to viscosity solutions of second order partial differential equations.

Bull. Amer. Math. Soc. (N.S.) **27** (1992), *no. 1,* 1–67.

This is a self-contained presentation and survey of results involving the powerful and conceptually simple concept of viscosity solutions. The concept allows one to establish uniqueness and existence results for rather general boundary problems for nonlinear partial differential equations of the form $F(x,u,Du,D^2u)=0$, where F satisfies the monotonicity condition $F(x,r,p,X)\le F(x,s,p,Y)$ for all x,p in $\mathbf{R}^N$ and for all $r\le s$, $X\le Y$ (in the sense of the usual partial order on the space of $N\times N$ symmetric matrices). The main idea is to relax the smoothness property expected from the classical solution, as follows: $(p,X)\in J^{2,+}(u,x)$ (second order superjet of u at x), provided that for $y\to x$ we have the inequality

$$u(y)\le u(x)+\langle p,y-x\rangle+\langle X(y-x),y-x\rangle+o(|y-x|^2).$$

The reverse inequality gives rise to $J^{2,-}$, the second-order subjet. A function u is a subsolution of $F=0$ provided u is upper semicontinuous and $F(x,u,p,X)\le 0$ for all X and for all $(p,X)\in J^{2,+}(u,x)$. With supersolutions defined in a similar manner, one can now proceed, somewhat like in the classical Perron method, to treat the boundary problems indicated above. The paper contains discussions of numerous important and difficult examples and 161 references. *P. Szeptycki*

References

1. A.D. Aleksandrov, *Almost everywhere existence of the second differential of a convex function and some properties of convex functions*, Leningrad. Univ. Ann. (Math. Ser.) **37** (1939), 3-35. (Russian)
2. A.D. Aleksandrov, *Uniqueness conditions and estimates for the solution of the Dirichlet problem*, Amer. Math. Soc. Transl., vol. 68, Amer. Math. Soc., Providence, RI, 1968, pp. 89-119.
3. A.D. Aleksandrov, *Majorization of solutions of second-order linear equations*, Amer. Math. Soc. Transl., vol. 68, Amer. Math. Soc., Providence, RI, 1968, pp. 120-143.
4. R. Alziary de Roquefort, *Jeux de poursuite et approximation des fonctions valeur*, Thèse, Universitè de Paris-Dauphine, 1990; Modél. Math. Anal. Num. **25** (1991), 535-560. MR1111654
5. S. Aizawa and Y. Tomita, *On unbounded viscosity solutions of a semilinear second order elliptic equation*, Funkcial. Ekvac. **31** (1988), 141-160. MR0951771
6. I. Ja. Bakelman, *On the theory of quasilinear elliptic equations*, Sibirsk. Mat. Z. **2** (1961), 179-186. MR0126604
7. V. Barbu, *Hamilton-Jacobi equations and nonlinear control problems*, preprint. MR0864766
8. V. Barbu, *The dynamic programming equation for the time optimal control problem in infinite dimensions*, SIAM J. Control Optim. **29** (1991), 445-456. MR1092737
9. V. Barbu, E. N. Barron, and R. Jensen, *Necessary conditions for optimal control in Hilbert spaces*, J. Math. Anal. Appl. **133** (1988), 151-162. MR0949325

10. V. Barbu and G. Da Prato, *Hamilton-Jacobi equations in Hilbert spaces*, Research Notes in Math., vol. 86, Pitman, Boston, MA, 1983. MR0704182
11. M. Bardi, *An asymptotic formula for the Green's function of an elliptic operator*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1987), 569-612. MR0963490
12. M. Bardi and L. C. Evans, *On Hopf's formulas for solutions of Hamilton-Jacobi equations*, Nonlinear Anal. Theory Methods Appl. **8** (1984), 1373-1381. MR0764917
13. M. Bardi and B. Perthame *Exponential decay to stable states in phase transitions via a double log-transformation*, Comm. Partial Differential Equations **15** (1990), 1649-1669. MR1080616
14. G. Barles, *Existence results for first-order Hamilton-Jacobi equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **1** (1984), 325-340. MR0779871
15. G. Barles, *Remarques sur des résultats d'existence pour les équations de Hamilton-Jacobi du premier ordre*, Ann. I. H. Poincaré Anal. Non Linéaire **2** (1985), 21-33. MR0781590
16. G. Barles, *Remark on a flame propagation model*, Rapport INRIA, no. 464, 1985.
17. G. Barles, *Interior gradient bounds for the mean curvature equation by viscosity solution methods*, Differential Integral Equations **4** (1991), 263-275. MR1081183
18. G. Barles, *A weak Bernstein method for fully nonlinear elliptic equations*, Differential Integral Equations **4** (1991), 241-262. MR1081182
19. G. Barles, *Uniqueness and regularity results for first-order Hamilton-Jacobi equations*, Indiana Univ. Math. J. **39** (1990), 443-466. MR1089047
20. G. Barles, *Regularity results for first-order Hamilton-Jacobi equations*, Differential Integral Equations **3** (1990), 103-125. MR1014729
21. G. Barles, *Fully nonlinear Neumann type boundary conditions for second-order elliptic and parabolic equations*, J. Differential Equations (to appear). MR1249178
22. G. Barles, L. Bronsard, and P. E. Souganidis, *Front propagation for reaction-diffusion equations of bistable type*, Analyse Non Linéaire (to appear). MR1191007
23. G. Barles, L. C. Evans, and P. E. Souganidis, *Wave front propagation for reaction-diffusion systems of PDE's*, Duke Univ. Math. J. **61** (1990), 835-858. MR1084462
24. G. Barles and P. L. Lions, *Fully nonlinear Neumann type boundary conditions for first-order Hamilton-Jacobi equations*, Nonlinear Anal. Theory Methods Appl. **16** (1991), 143-153. MR1090787
25. G. Barles and B. Perthame, *Discontinuous solutions of deterministic optimal stopping-time problems*, Modèl. Math. Anal. Num. **21** (1987), 557-579. MR0921827
26. G. Barles and B. Perthame, *Exist time problems in optimal control and the vanishing viscosity method*, SIAM J. Control Optim. **26** (1988), 1133-1148. MR0957658
27. G. Barles and B. Perthame, *Comparison principle for Dirichlet-type H. J. equations and singular perturbations of degenerate elliptic equations*, Appl. Math. Optim. **21** (1990), 21-44. MR1014943
28. G. Barles and P. E. Souganidis, *Convergence of approximation schemes for fully nonlinear second order equations*, Asymp. Anal. **4** (1991), 271-283. MR1115933
29. E. N. Barron, *Differential games with maximum cost*, Nonlinear Anal. Theory Methods Appl. **14** (1990), 971-989. MR1058417
30. E. N. Barron and H. Ishii, *The Bellman equation for minimizing the maximum cost*, Nonlinear Anal. Theory Methods Appl. **13** (1989), 1067-1090. MR1013311
31. E. N. Barron and R. Jensen, *The Pontryagin maximum principle from dynamic programming and viscosity solutions to first-order partial differential equations*, Trans. Amer. Math. Soc. **298** (1986), 635-641. MR0860384
32. E. N. Barron and R. Jensen, *Generalized viscosity solutions for Hamilton-Jacobi equations with time-measurable Hamiltonians*, J. Differential Equations **68** (1987), 10-21. MR0885811

33. E. N. Barron and R. Jensen, *Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians*, Comm. Partial Differential Equations **15** (1990), 1713-1742. MR1080619
34. E. N. Barron and R. Jensen, *Optimal control and semicontinuous viscosity solutions*, Proc. Amer. Math. Soc. **113** (1991), 397-402. MR1076572
35. E. N. Barron and R. Jensen, *A stochastic control approach to the pricing of options*, Math. Operations Research **15** (1990), 49-79. MR1038235
36. S. Benton, *The Hamilton-Jacobi equation: A global approach*, Academic Press, New York, 1977. MR0442431
37. A. D. Bony, *Principe du maximum dans les espaces de Sobolev*, C. R. Acad. Sci. Paris Ser. A **265** (1967), 333-336. MR0223711
38. H. Brezis, *Operateurs Maximaux Monotones et semigroupes de contractions dans les espaces de Hilbert*, North Holland, Amsterdam, 1973. MR0348562
39. L. Caffarelli, *Interior a priori estimates for solutions of fully non-linear equations*, Ann. of Math. (2) **130** (1989), 180-213. MR1005611
40. P. Cannarsa and M. Soner, *On the singularities of the viscosity solutions to Hamilton-Jacobi-Bellman equations*, Indiana Univ. Math. J. **46** (1987), 501-524. MR0905608
41. P. Cannarsa and M. Soner, *Generalized one-sided estimates for solutions of Hamilton-Jacobi equations and applications*, Nonlinear Anal. Theory Methods Appl. **13** (1989), 305-323. MR0986450
42. P. Cannarsa, F. Gozzi, and M. Soner, *A boundary value problem for Hamilton-Jacobi equations in Hilbert spaces*, Appl. Math. Opt. **24** (1991), 197-220. MR1118698
43. I. Cappuzzo-Dolcetta and L. C. Evans, *Optimal switching for ordinary differential equations*, SIAM J. Optim. Control **22** (1988), 1133-1148. MR0728678
44. I. Cappuzzo-Dolcetta and P. L. Lions, *Viscosity solutions of Hamilton-Jacobi-equations and state constraints*, Trans. Amer. Math. Soc. **318** (1990), 643-683. MR0951880
45. Y. G. Chen, Y. Giga, and S. Goto, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equations*, J. Differential Geom. **33** (1991), 749-786. MR1100211
46. M. G. Crandall, *Quadratic forms, semidifferentials and viscosity solutions of fully nonlinear elliptic equations*, Ann. I. H. Poincaré Anal. Non Linéaire **6** (1989), 419-435. MR1035337
47. M. G. Crandall, L. C. Evans, and P. L. Lions, *Some properties of viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **282** (1984), 487-502. MR0732102
48. M. G. Crandall and H. Ishii, *The maximum principle for semicontinuous functions*, Differential Integral Equations **3** (1990), 1001-1014. MR1073054
49. M. G. Crandall, H. Ishii, and P. L. Lions, *Uniqueness of viscosity solutions of Hamilton-Jacobi equations revisited*, J. Math. Soc. Japan **39** (1987), 581-596. MR0905626
50. M. G. Crandall and P. L. Lions, *Condition d'unicité pour les solutions generalisées des équations de Hamilton-Jacobi du premier ordre*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), 183-186. MR0610314
51. M. G. Crandall and P. L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc. **277** (1983), 1-42. MR0690039
52. M. G. Crandall and P. L. Lions, *Two approximations of solutions of Hamilton-Jacobi equations*, Math. Comp. **43** (1984), 1-19. MR0744921
53. M. G. Crandall and P. L. Lions, *On existence and uniqueness of solutions of Hamilton-Jacobi equations*, Nonlin. Anal. Theory Methods Appl. **10** (1986), 353-370. MR0836671

54. M. G. Crandall and P. L. Lions, *Unbounded viscosity solutions of Hamilton-Jacobi equations*, Illinois J. Math. **31** (1987), 665-688. MR0909790
55. M. G. Crandall and P. L. Lions, *Hamilton-Jacobi equations in infinite dimensions, Part* I. *Uniqueness of viscosity solutions*, J. Func. Anal. **62** (1985), 379-396; *Part* II. *Existence of viscosity solutions* **65** (1986), 368-405; *Part* III **68** (1986), 214-247; **Part** IV. *Unbounded linear terms* **90** (1990), 237-283; *Part* V. B-*continuous solutions*, **97** (1991), 417-465. MR1052335
56. M. G. Crandall and P. L. Lions, *Quadratic growth of solutions of fully nonlinear second order equations on* $\mathbb{R}^n$, Differential Integral Equations **3** (1990), 601-616. MR1044206
57. M. G. Crandall, P. L. Lions, and P. E. Souganidis, *Maximal solutions and universal bounds for some quasilinear evolution equations of parabolic type*, Arch. Rat. Mech. Anal. **105** (1989), 163-190. MR0968459
58. M. G. Crandall and R. Newcomb, *Viscosity solutions of Hamilton-Jacobi equations at the boundary*, Proc. Amer. Math. Soc. **94** (1985), 283-290. MR0784180
59. M. G. Crandall, R. Newcomb, and Y. Tomita, *Existence and uniqueness of viscosity solutions of degenerate quasilinear elliptic equations in* $\mathbb{R}^n$, Appl. Anal. **34** (1989), 1-23. MR1333953
60. P. Dupuis and H. Ishii, *On oblique derivative problems for fully nonlinear second order elliptic equations on nonsmooth domains*, Nonlinear Anal. Theory Methods Appl. **12** (1991), 1123-1138. MR1082287
61. P. Dupuis and H. Ishii, *On oblique derivative problems for fully nonlinear second order elliptic PDE's on domains with corners*, Hokkaido Math. J. **20** (1991), 135-164. MR1096165
62. P. Dupuis, H. Ishii, and H. M. Soner, *A viscosity solution approach to the asymptotic analysis of queuing systems*, Ann. Probab. **18** (1990), 226-255. MR1043946
63. H. Engler and S. Lenhart, *Viscosity solutions for weakly coupled systems of Hamilton-Jacobi equations*, Proc. London Math. Soc. **63** (1991), 212-240. MR1105722
64. L. C. Evans, *A convergence theorem for solutions of nonlinear second order elliptic equations*, Indiana Univ. Math. J. **27** (1978), 875-887. MR0503721
65. L. C. Evans, *On solving certain nonlinear differential equations by accretive operator methods*, Israel J. Math. **36** (1980), 225-247. MR0597451
66. L. C. Evans, *The perturbed test function technique for viscosity solutions of partial differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **111** (1989), 359-375. MR1007533
67. L. C. Evans and R. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Math., CRC Press, Ann Arbor, 1992. MR1158660
68. L. C. Evans and H. Ishii, *A pde approach to some asymptotic problems concerning random differential equations with small noise intensities*, Ann. I. H. Poincaré Anal. Non Linéaire **2** (1985), 1-20. MR0781589
69. L. C. Evans, M. Soner, and P. E. Souganidis, *The Allen-Cahn equation and generalized motion by mean curvature*, preprint. MR1177477
70. L. C. Evans and P. E. Souganidis, *Differential games and representation formulas for solutions of Hamilton-Jacobi-Isaacs equations*, Indiana Univ. Math. J. **33** (1984), 773-797. MR0756158
71. L. C. Evans and P. E. Souganidis, *A PDE approach to geometric optics for semilinear parabolic equations*, Indiana Univ. Math. J. **38** (1989), 141-172. MR0982575
72. L. C. Evans and J. Spruck, *Motion of level sets by mean curvature*. I, J. Differential Geom. **33** (1991), 635-681. MR1100206
73. M. Falcone, *A numerical approach to the infinite horizon problem of deterministic control theory*, Appl. Math. Optim. **15** (1987), 1-13. MR0866164

74. H. Federer, *Geometric measure theory*, Springer, New York, 1969. MR0257325
75. W. H. Fleming and R. Rishel, *Determinisitic and stochastic optimal control*, Springer-Verlag, New York, 1975. MR0454768
76. W. H. Fleming and P. E. Souganidis, *PDE-viscosity solution approach to some problems of large deviations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **13** (1986), 171-192. MR0876121
77. W. H. Fleming and P. E. Souganidis, *On the existence of value functions of two player, zero-sum stochastic differential games*, Indiana Univ. Math. J. **38** (1989), 293-314. MR0997385
78. H. Frankowska, *On the single-valuedness of Hamilton-Jacobi operators*, Nonlin. Anal. Theory Methods Appl. **10** (1986), 1477-1483. MR0869555
79. Y. Giga, S. Goto, H. Ishii, and M.-H. Sato, *Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains*, Indiana U. Math. J. **40** (1991), 443-470. MR1119185
80. Y. Giga, S. Goto, and H. Ishii, *Global existence of weak solutions for interface equations coupled with diffusion equations*, SIAM J. Math. Anal, (to appear). MR1166559
81. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd Ed., Springer-Verlag, New York, 1983. MR0737190
82. H. Ishii, *Remarks on existence and uniqueness of viscosity solutions of Hamilton-Jacobi equations*, Bull. Fac. Sci. Eng. Chuo Univ. **26** (1983), 5-24. MR0742691
83. H. Ishii, *Uniqueness of unbounded viscosity solutions of Hamilton-Jacobi equations*, Indiana Univ. Math. J. **33** (1984), 721-748. MR0756156
84. H. Ishii, *Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets*, Bull. Fac. Sci. Eng. Chuo Univ. **28** (1985), 33-77. MR0845397
85. H. Ishii, *Existence and uniqueness of solutions of Hamilton-Jacobi equations*, Funkcial. Ekvac. **29** (1986), 167-188. MR0877427
86. H. Ishii, *Perron's method for Hamilton-Jacobi equations*, Duke Math. J. **55** (1987), 369-384. MR0894587
87. H. Ishii, *A simple direct proof for uniqueness for solutions of the Hamilton-Jacobi equations of eikonal type*, Proc. Amer. Math. Soc **100** (1987), 247-251. MR0884461
88. H. Ishii, *Representation of solutions of Hamilton-Jacobi equations*, Nonlin. Anal. Theory Methods Appl. **12** (1988), 121-146. MR0926208
89. H. Ishii, *On uniqueness and existence of viscosity solutions of fully nonlinear second-order elliptic PDE's*, Comm. Pure Appl. Math. **42** (1989), 14-45. MR0973743
90. H. Ishii, *A boundary value problem of the Dirichlet type for Hamilton-Jacobi equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **16** (1989), 105-135. MR1056130
91. H. Ishii, *Fully nonlinear oblique derivative problems for nonlinear second-order elliptic PDE's*, Duke Math. J. **62** (1991), 663-661. MR1104812
92. H. Ishii, *Perron's method for monotone systems of second-order elliptic PDEs*, Diff. Integral Equations **5** (1992), 1-24. MR1141724
93. H. Ishii and S. Koike, *Remarks on elliptic singular perturbation problems*, Appl. Math. Optim. **23** (1991), 1-15. MR1076052
94. H. Ishii and S. Koike, *Viscosity solutions of a system of nonlinear second-order elliptic PDEs arising in switching games*, Funkcial. Ekvac. **34** (1991), 143-155. MR1116886
95. H. Ishii and S. Koike, *Viscosity solutions for monotone systems of second-order elliptic PDE's*, Comm. Partial Differential Equations **16** (1991), 1095-1128. MR1116855
96. H. Ishii and P. L. Lions, *Viscosity solutions of fully nonlinear second-order elliptic partial differential equations*, J. Differential Equations **83** (1990), 26-78. MR1031377
97. H. Ishii and N. Yamada, *A remark on a system of inequalities with bilateral obstacles*, Nonlinear Anal. Theory Methods Appl. **13** (1989), 1295-1301. MR1021407

98. A. V. Ivanov, *Quasilinear degenerate and nonuniformly elliptic equations of second order*, Proc. Steklov Inst. Math. **1** (1984), 1-287. MR0753230
99. S. Koike, *On the rate of convergence of solutions in singular perturbation problems*, J. Math. Anal. Appl. *157* (1991), 277-292. MR1109454
100. S. Koike, *An asymptotic formula for solutions of Hamilton-Jacobi-Bellman equations*, Nonlinear Anal. Theory Methods Appl. **11** (1987), 429-436. MR0881728
101. R. Jensen, *The maximum principle for viscosity solutions of fully nonlinear second order partial differential equations*, Arch. Rat. Mech. Anal. **101** (1988), 1-27. MR0920674
102. R. Jensen, *Uniqueness criteria for viscosity solutions of fully nonlinear elliptic partial differential equations*, Indiana Univ. Math. J. **38** (1989), 629-667. MR1017328
103. R. Jensen, *Local quadratic growth estimates for elliptic obstacle problems*, preprint.
104. R. Jensen, P. L. Lions, and P. E. Souganidis, *A uniqueness result for viscosity solutions of second order fully nonlinear partial differential equations*, Proc. Amer. Math. Soc. **102** (1988), 975-978. MR0934877
105. R. Jensen and P. E. Souganidis, *regularity result for viscosity solutions of Hamilton-Jacobi equations in one space dimension*, Trans. Amer. Math. Soc. **301** (1987), 137-147. MR0879566
106. J. J. Kohn and L. Nirenberg, *Degenerate elliptic-parabolic equations of second order*, Comm. Pure Appl. Math. **20** (1967), 797-872. MR0234118
107. S. N. Kruzkov, *First order quasilinear equations in several independent variables*, Math. USSR-Sb. **10** (1970), 217-243. MR0267257
108. N. V. Krylov, *Controlled diffusion processes*, Springer-Verlag, Berlin, 1980. MR0601776
109. N. V. Krylov, *Nonlinear elliptic and parabolic equations of second order*, D. Reidel Publishing Co., Boston, MA, 1987. MR0815513
110. J. M. Lasry and P. L. Lions, *A remark on regularization in Hilbert spaces*, Israel J. Math **55** (1986), 257-266. MR0876394
111. J. M. Lasry and P. L. Lions, *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints*, I, *The model problem*, Math. Ann. **283** (1989), 583-630. MR0990591
112. S. Lenhart, *Integro-differential operators associated with diffusion processes with jumps*, Appl. Math. Optim. **9** (1982), 177-191. MR0680451
113. S. Lenhart, *Viscosity solutions for weakly coupled systems of first order PDEs*, J. Math. Anal. Appl. **131** (1988), 180-193. MR0934440
114. S. Lenhart and N. Yamada, *Perron's method for viscosity solutions associated with piecewise-deterministic processes*, Funkcial. Ekvac. (to appear). MR1116888
115. S. Lenhart and N. Yamada, *Viscosity solutions associated with switching game for piecewise-deterministic processes*, preprint. MR1274894
116. P. L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Mathematics 69, Pitman, Boston, MA, 1982. MR0667669
117. P. L. Lions, *On the Hamilton-Jacobi-Bellman equations*, Acta Appl. **1** (1983), 17-41. MR0702045
118. P. L. Lions, *Some recent results in the optimal control of diffusion processes: Stochastic analysis*, Proceedings of the Taniguchi International Symposium on Stochastic Analysis (Katata and Kyoto 1982) Kinokuniya, Tokyo, 1984. MR0780764
119. P. L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. Part 1: The dynamic programming principle and applications and Part 2: Viscosity solutions and uniqueness*, Comm. Partial Differential Equations **8** (1983), 1101-1174 and 1229-1276. MR0709162
120. P. L. Lions, *A remark on the Bony maximum principle*, Proc. Amer. Math. Soc. **88**

(1983), 503-508. MR0699422
121. P. L. Lions, *Neumann type boundary conditions for Hamilton-Jacobi equations*, Duke J. Math. **52** (1985), 793-820. MR0816386
122. P. L. Lions, *Regularizing effects for first-order Hamilton-Jacobi equations*, Applic. Anal. **20** (1985), 283-308. MR0814955
123. P. L. Lions, *Equations de Hamilton-Jacobi et solutions de viscositè*, Colloque De Giorgi, Paris, 1983; Pitman, London, 1985. MR0909709
124. P. L. Lions, *Some properties of the viscosity semigroups of Hamilton-Jacobi equations*, Nonlinear Differential Equations and Applications, Pitman, London, 1988. MR0667669
125. P. L. Lions, *Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. Part* I: *The case of bounded stochastic evolutions*, Acta Math. **161** (1988), 243-278. MR0971797
126. P. L. Lions, *Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. Part* II: Optimal Control of Zakai's Equation, Stochastic Partial Differential Equations and Applications II, Lecture Notes in Math., vol. 1390, Springer, Berlin, 1989. (G. DuPrato and L. Tubaro, eds.) Proceedings of the International Conference on Infinite Dimensional Stochastic Differential Equations, Trento, Springer, Berlin, 1988. MR1019600
127. P. L. Lions, *Viscosity solutions of fully nonlinear second-order equations and optimal stochastic control in infinite dimensions. Part* III: *Uniqueness of viscosity solutions of general second order equations*, J. Funct. Anal. **86** (1989), 1-18. MR1013931
128. P. L. Lions, G. Papanicolau, and S. R. S. Varadhan, *Homogenization of Hamilton-Jacobi equations*, preprint. MR0909709
129. P. L. Lions and B. Perthame, *Remarks on Hamilton-Jacobi-Equations with measurable time-dependent Hamiltonians*, Nonlinear Anal. Theory Methods Appl. **11** (1987), 613-622. MR0886652
130. P. L. Lions and M. Nisio, *A uniqueness result for the semigroup associated with the Hamilton-Jacobi-Bellman operator*, Proc. Japan Acad. **58** (1982), 273-276. MR0682680
131. P. L. Lions and Rochet, *Hopf formula and multitime Hamilton-Jacobi equations*, Proc. Amer. Math. Soc. **90** (1980), 79-84. MR0813815
132. P. L. Lions and P. E. Souganidis, *Viscosity solutions of second-order equations, stochastic control and stochastic differential games*, Stochastic Differential Systems, Stochastic Control Theory and Applications (W. H. Fleming and P. L. Lions, eds.), IMA Vol. Math. Appl., vol. 10, Springer, Berlin, 1988. MR0934730
133. P. L. Lions and P. E. Souganidis, *Convergence of MUSCL type methods for scalar conservation laws*, C. R. Acad. Sci. Paris Sér. A **311** (1990), 259-264. MR1071622
134. P. L. Lions, P. E. Souganidis, and J. Vazquez, *The relation between the porous medium equation and the eikonal equation in several space dimensions*, Rev. Math. Ibero. **3** (1987), 275-310. MR0996819
135. F. Mignot, *Contrôle optimal dans les inéquations variationelles elliptiques*, J. Funct. Anal. **22** (1976), 130-185. MR0423155
136. R. Newcomb, *Existence and correspondence of value functions and viscosity solutions of Hamilton-Jacobi equations* Univ. of Wisconsin-Madison, 1988. MR2636727
137. D. Nunziante, *Uniqueness of viscosity solutions of fully nonlinear second-order fully nonlinear parabolic equations with discontinuous time-dependence*, Differential Integral Equations **3** (1990), 77-91. MR1014727
138. D. Nunziante, *Existence and uniqueness of unbounded viscosity solutions of parabolic equations with discontinuous time-dependence*, preprint. MR1167420
139. O. A. Oleĭnik and E. V. Radekvic, *Second order equations with nonnegative charac-

teristic form, American Mathematical Society, Providence, RI, 1973. MR0457908

140. S. Osher and J. Sethian, *Fronts propagating with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations*, J. Comp. Physics **79** (1988), 12-49. MR0965860
141. B. Perthame, *Perturbed dynamical systems with an attracting singularity and weak viscosity limits in H. J. equations*, Trans. Amer. Math. Soc. **317** (1990), 723-747. MR0943607
142. B. Perthame and R. Sanders, *The Neumann problem for fully nonlinear second order singular perturbation problems*, SIAM J. Math. Anal. **19** (1988), 295-311. MR0930028
143. C. Pucci, *Su una limitazone per soluzioni di equazioni ellittche*, Boll. Un. Mat. Ital. (6) **21** (1966), 228-233. MR0204838
144. E. Rouy and A. Tourin, *A viscosity solution approach to shape—from shading*, preprint. MR1163361
145. W. Rudin, *Real and complex analysis*, 3rd ed., Mc Graw-Hill, New York, 1987. MR0924157
146. M. Sato, *Comparison principle for singular degenerate elliptic equations on unbounded domains*, Proc. Japan Acad. Ser. A **66** (1990), 252-256. MR1077608
147. N. Yamada, *Viscosity solutions for a system of inequalities with biblateral obstacles*, Funkcial. Ekvac. **30** (1987), 417-425. MR0927191
148. A. Sayah, *Equations d'Hamilton-Jacobi du premier ordre avec termes intégro differentiels. Partres* I *and* II, Comm. Partial Differential Equations **16** (1991), 1057-1221.
149. M. Soner, *Optimal control with state-space constraint* I, SIAM J. Control Optim. **24** (1986), 552-562. MR0838056
150. M. Soner, *Optimal control with state-space constraint* II, SIAM J. Control Optim. **24** (1986), 1110-1122. MR0861089
151. M. Soner, *On the Hamilton-Jacobi-Bellman equations in Banach spaces*, J. Optim. Theory Appl. **57** (1988), 121-141. MR0944588
152. M. Soner, *Motion of a set by the curvature of its mean boundary*, preprint. MR1204331
153. P. E. Souganidis, *Max-Min representations and product formulas for the viscosity solutions of Hamilton-Jacobi equations with applications to differential games*, Nonlin. Anal. Theory Methods Appl. **9** (1985), 217-257. MR0784388
154. P. E. Souganidis, *Existence of viscosity solutions of Hamilton-Jacobi equations*, J. Differential Equations **56** (1985), 345-390. MR0780496
155. P. E. Souganidis, *Approximation schemes for viscosity solutions of Hamilton-Jacobi equations*, J. Differential Equations **59** (1985), 1-43. MR0803085
156. P. E. Souganidis, *A remark about viscosity solutions at the boundary*, Proc. Amer. Math. Soc. **96** (1986), 323-330. MR0818466
157. D. Tataru, *Viscosity solutions of Hamilton-Jacobi equations with unbounded nonlinear terms*, J. Math. Anal. Appl. **163** (1992), 345-392. MR1145836
158. N. S. Trudinger, *Comparison principles and pointwise estimates for viscosity solutions of second order elliptic equations*, Rev. Mat. Ibero. **4** (1988), 453-468. MR1048584
159. N. S. Trudinger, *Hölder gradient estimates for fully nonlinear elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A **108** (1988), 57-65. MR0931007
160. N. S. Trudinger, *On the twice differentiability of viscosity solutions of nonlinear elliptic equations*, Bull. Austral. Math. Soc. **39** (1989), 443-447. MR0995142
161. J. L. Vila and T. Zariphopolou, *Optimal consumption and portfolio choice with borrowing constraints*, preprint. MR1605885
162. T. Zariphopolou, *Investment-consumption model with transaction costs and Markov-*

Chain parameters, SIAM J. Control Optim. **30** (to appear, 1992). MR1160145

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.